An NP-complete Problem for the Abelian Sandpile Model

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A configuration of the Abelian sandpile model is called *recurrent* if grains of sand can be added to it so that the resulting configuration relaxes to the original one; else it is called *transient*. Given a transient configuration, a recurrent configuration results after adding enough grains of sand. In the case of a three-dimensional sandpile model, it is shown in this paper that the problem of finding the minimal number of grains that have to be added to a given transient configuration to get a recurrent one, interpreted as the distance of this configuration to the set of recurrent configurations, is NP-complete.

1. Introduction

The Abelian sandpile model is the standard model for self-organized criticality, a concept introduced by Bak, Tang, and Wiesenfeld in [1]: When adding grains of sand one after another to a sandpile, grains of sand will fall from one site to neighboring sites if there are too many grains on it. On the border of the sandpile, grains will fall down from the pile. It was found that the frequency distribution of the size of avalanches the addition of a single grain of sand can cause seems to follow a power law quite similar to the power law describing the magnitudes of earthquakes.

Starting with an empty sandpile and adding grains of sand to randomly chosen sites, one will eventually reach configurations that can reappear some time in the future; these are called *recurrent* configurations and possess many nice algebraic properties [2]. Initially each grain of sand added to the nearly empty sandpile gets us nearer to a recurrent configuration, later on this is not the case anymore. While it would be interesting to study a diagram showing the minimal number of grains needed to reach a recurrent configuration *versus* the number of already added grains, it is often very time-consuming to compute this number for a given configuration. In fact, we show that this problem, which we will call DIST, is NP-complete if we consider a three-dimensional sandpile model.

First of all, we give basic definitions for the sandpile model and state some useful facts concerning recurrent configurations. Next, we formulate the problem investigated in this paper. After proving some facts concerning avalanches caused by the addition of a fixed number of grains we show how to construct a configuration c_G for a given graph G, such that the minimal number of grains that have to be added to c_G is the size of a minimal vertex cover of G. After proving this, we get that the considered problem DIST is NP-complete. Finally, we discuss problems arising in the two-dimensional case.

2. Basic definitions

2.1 Three-dimensional sandpile model

The three-dimensional sandpile model is given by a cell space $Z = \mathbf{p} \times \mathbf{q} \times \mathbf{r}$, where $\mathbf{p}$ means the set of natural numbers from 1 to p, and the toppling function F: $\mathbb{Z}^Z \times Z \longrightarrow \mathbb{Z}^Z$,

$$F(c, z)(z') = \begin{cases} c(z) - 6 & \text{if } z = z' \\ c(z) + 1 & \text{if } \|z - z'\|_1 = 1 \\ c(z) & \text{else.} \end{cases}$$

Note that this function is not number-conserving, since there are losses at the border of the cell space.

Given a configuration $c_0 \in \mathbf{Z}^Z$, a cell z satisfying $c_0(z) \geq 6$ is chosen (if such a cell exists), and the configuration $c_1 = F(c, z)$ is computed (we will say the cell *toppled*). This is repeated, until a configuration c_k is reached which satisfies $\forall z \in Z : c_k(z) < 6$. This process is called a *relaxation* of c, abbreviated Rel(c), and the configuration c_k is denoted c_{rel}. (That this configuration exists and is unique is proven, for example, in [3].) A configuration satisfying $\forall z \in Z : c(z) < 6$ is called a *stable* configuration.

The toppling of a cell can be interpreted this way: Consider a sandpile where each cell z contains $c(z)$ grains of sand. If there are more than five grains of sand in cell z, then six grains of sand fall to the neighboring cells, or off the sandpile, if the cell is set at the boundary of the cell space.

2.2 Recurrent and transient configurations

Consider the following process: Starting from a stable configuration c^0, a cell z^0 is picked at random and its value is increased by one to get the configuration c'^1. Afterwards, $c^1 = c'^1_{\text{rel}}$ is computed. If repeated infinitely, there are stable configurations that can occur several times; these are called *recurrent* configurations. Nonrecurring configurations are called *transient*. (Please note that the configuration we get by adding n grains of sand to a given configuration c and afterwards relaxing the resulting configuration is the same as the configuration we get by adding these n grains of sand one after another and relaxing each resulting configuration before adding the next grain of sand.)

If this process is considered a Markov chain with the set of stable configurations as state space, it can easily be shown that the configurations that are recurrent respectively transient according to our definition are recurrent respectively transient according to the usual definition for Markov chains.

Consider the configuration b satisfying

$$\forall z \in Z : b(z) = 6 - |\{z' \in Z : \|z - z'\|_1 = 1\}; \quad (1)$$

this is called the *burning* configuration and satisfies:

1. c is recurrent $\iff$ $(c + b)_{\text{rel}} = c$
2. c is recurrent $\iff$ During $\text{Rel}(c + b)$, each cell topples once
3. $(c + e)_{\text{rel}}$ is recurrent $\iff$ During $\text{Rel}((c + b)_{\text{rel}} + e)$, all cells topple which did not topple during $\text{Rel}(c + b)$.

For proofs, see [4].

3. The DIST problem

Consider a transient configuration c of a $p \times q \times r$ sandpile model. We know that, if we add grains of sand to randomly chosen cells and relax the resulting configurations, sooner or later we get a recurrent configuration.

Since "sooner or later" is not a very exact statement, we want to know how many grains of sand have to be added to get a recurrent configuration.

Formally, this problem can be described as finding the minimal number k such that there exists a configuration e satisfying $\sum_{z \in Z} e(z) = k$ and $(c + e)_{\text{rel}}$ is recurrent.

We interpret this number $\text{dist}(c)$ as the distance of c to the set of recurrent configurations, and therefore call the problem of determining this number DIST.

3.1 Claim

DIST is NP-complete.

To prove this claim, we first need some facts concerning the behavior of avalanches that occur when one grain of sand is added to a stable configuration.

The NP-completeness proof uses the fact that the problem VERTEX COVER is NP-complete (cf. [5]). We will describe the construction of a configuration c_G for a given graph G, such that $\text{dist}(c_G) = \text{minVC}(G)$ ($\text{minVC}(G)$ denotes the size of a minimum vertex cover of G). It will be easy to see that $\text{dist}(c_G) \leq \text{minVC}(G)$ holds, and using the facts about avalanches we will prove first, we show that $\text{dist}(c_G) = \text{minVC}(G)$. Thus, DIST is NP-complete.

4. Adding grains of sand to blocks

Lemma 1. Consider a configuration c of the sandpile model where exactly one cell $z \in Z$ contains six grains of sand and all other cells less than six.

During the relaxation, for each $l \geq 1$ the following statement holds: If there is a cell that topples l times, z is the first cell to topple l times.

Proof. Consider the first cell z' to topple l times; before, this cell has already toppled $l-1$ times, and all neighbors of z' have at most toppled $l-1$ times. Thus, z' has lost $6(l-1)$ grains and received at most $6(l-1)$ grains from its neighbors; therefore, z' can contain at most $c(z')$ grains of sand. Since z' can topple and z is the only cell satisfying $c(z) = 6$, we get $z' = z$. ■

Lemma 2. Consider a configuration c and a block $B \subset Z$ of dimension $d_1 \times d_2 \times d_3$ such that the following statements hold.

1. There is exactly one cell $z \in Z$ satisfying $c(z) = 6$.
2. All cells $z' \neq z$ satisfy $c(z') < 6$.
3. $z \in B$.
4. All cells z' satisfying $z' \notin B$, $\exists \tilde{z} \in B$: z' and $\tilde{z}$ are neighbors, contain at most four grains of sand.

Then all cells that topple during the relaxation of c are contained in B.

Proof. For each cell $z' \in B$, consider the distance of z' to the upper border of B, where the cells whose neighbor above them is not contained in B have the distance 1; all cells above B shall have the distance 0.

Assume that there is a cell either contained in B or just above the upper border with distance k to the upper border of B that topples $k+1$ times during the relaxation of c.

Let z' then be the first cell to topple more often than its distance to the upper border of B is.

Assume that z' does not lie in B; then the distance of z' is 0. When z' topples for the first time, the cell above z' has not toppled yet, and neither have the cells neighboring z' in the same plane (else, z' could not be the first cell with distance 0 to topple one time). Only the cell below z' can have toppled, but at most once, since else the cell below z' would have toppled two times while having distance 1 before z' toppled one time having the distance 0. Further, if the cell below z' has toppled before, then this cell belongs to B and we know that $c(z') \leq 4$ holds. Therefore, z' cannot contain more than five grains of sand when it topples for the first time. This is a contradiction.

Assume $z' \in B$ and the distance of z' is l. When z' topples for the $l+1$st time, the cell above z' has toppled at most $l-1$ times, the neighbors in the same plane as z' have toppled at most l times, and the cell below z' at most $l+1$ times. So, z' has lost $6l$ grains of sand and gained at most $6l$ grains of sand. Since z' can topple, it follows that $c(z') = 6$ and therefore $z' = z$. Since $z' = z$, we know from Lemma 1 that the cell below z can only have toppled at most l times before z has toppled $l+1$ times. Therefore, z has gained at most $6l - 1$ grains of sand while having lost $6l$ grains of sand. Thus, z can contain at most five grains of sand and therefore cannot topple. This also is a contradiction.

It follows that no cell can topple more often than its distance from the upper border of B, and therefore no cell above B can topple during the relaxation.

Analogous arguments hold for the lower, left, right, frontal and rearward borders of B, and therefore no cell outside B can topple during the relaxation of c. ■

Lemma 3. Consider a configuration c, a subset $B \subseteq Z$, and a number l, such that the following statement holds: All cells z for which there exists a cell $z' \in B$ (not necessarily $z' \neq z$) such that $\|z - z'\|_{\max} \leq 2l$ satisfy $c(z) < 5$.

Let c' be a configuration that we get by adding l grains of sand to cells in B.

Then no cell z which satisfies $\|z - z'\|_{\max} \geq 2l$ topples during the relaxation of c'. (For the rest of this paper, we will call $\|z - z'\|_{\max}$ the *distance* from z to z'.)

Proof. By induction, we will show that the following two statements hold.

- If we add k grains to cells in B, $1 \leq k \leq l$, no cell having a distance greater than $2(k-1)$ from a cell in B can topple during the relaxation.
- If we add k grains to cells in B, $1 \leq k \leq l$, all cells containing five grains of sand lie in disjoint cubes $B_1, \ldots, B_p$ of lengths $k_1, \ldots, k_p$, each of which contains a cell at least one grain of sand has been added to and which satisfy $\sum_{i=1}^{p} k_i \leq 2k$.

Suppose we only add one grain of sand to B. Since all cells in B contain less than five grains of sand, the cell to which we added the grain of sand cannot topple. Therefore, no cell z with a distance greater than two to a cell in B can topple. Further, all cells that contain five grains of sand lie in a block of dimensions $2 \times 2 \times 2$ containing a cell a grain of sand has been added to.

Suppose the two given statements hold for some $k < l$.

Now, add one more grain of sand to a cell contained in B. If the cell contained at most four grains of sand, it is easy to see that the two statements still hold for $k + 1$. If the cell contained five grains of sand, it is contained in a block B_1 as specified. Now either all cells neighboring the border of B_1 contain no more than four grains of sand, or another block B_2 is directly adjacent to B_1. These two blocks are contained in a greater cube B_1' of length $k_1 + k_2$.

Now, again either all cells neighboring the border of B_1' contain no more than four grains of sand, or another block B_3 is directly adjacent to B_1'. Again, these blocks are contained in a larger cube B_2' of length $k_1 + k_2 + k_3$.

This procedure is repeated until all cells adjacent to B_j' contain no more than four grains of sand. The size of B_j' then is no more than $\sum_{i=1}^{p} k_i \leq 2k$.

According to Lemma 2, no cell outside this block can topple during the relaxation. Therefore, after adding $k + 1$ grains of sand, no cell having a distance greater than $2k$ from a cell in B can topple during the relaxation.

Consider the block B_j'. All cells containing four grains of sand before and five grains of sand after the relaxation are neighbors to cells in B_j' and are therefore contained in a cube B' of length $(\sum_{i=1}^{j+1} k_i) + 2$. All cells containing five grains either lie in B' or in blocks $B_h, j + 2 \leq h \leq p$ disjoint to the blocks contained in B_j'; the sum of the lengths of this block is no more than $(\sum_{i=1}^{j+1} k_i) + 2 + \sum_{i=j+2}^{p} k_i \leq 2k + 2$. Therefore, the second statement also holds.

From this, the claim directly follows. ■

5. NP-completeness of DIST

Let $G = (V, E)$ be a connected graph without loops. Let $n = |V|$, $m = |E|$, and $k = \text{minVC}(G)$. Further, let $V = \{u_1, \ldots, u_n\}$ and $E = \{e_1, \ldots, e_m\}$.

5.1 Idea

The idea of the construction is to get a configuration c_G, where all cells except the cells of m small areas (one area per edge of G; these shall be called *cover check areas*) topple during $\text{Rel}(c + b)$, and that there are n "fuzes" in c_G (one fuze per vertex), such that all cells in a cover check area corresponding to an edge $\{u, v\}$ topple if a grain of sand is added to the fuze corresponding to u or v. (We will call a fuze *incident* to an edge e if the corresponding vertex is incident to e.)

Therefore, we can find k fuzes such that all cells in all cover check areas topple during the relaxation of the configuration we get after adding one grain of sand to each fuze in $(c+b)_{\text{rel}}$ if we choose the fuzes corresponding to a minimum vertex cover. From that we know that the resulting configuration is recurrent and therefore $\text{dist}(c_G) \leq k$.

Further, we make sure that the only way to have all cover check areas topple adding at most n grains of sand is by adding grains of sand to cells near some fuzes, and so get the result $\text{dist}(c_G) = \text{minVC}(G)$.

5.2 Coarse description

We use a sandpile model whose grid is divided into $(n+1) \times m$ blocks, one block per vertex and edge and one additional block per edge which contains the cover check areas. The blocks on the lowest level shall correspond to the edge e_1, the leftmost blocks shall correspond to the vertex u_1. Then $n-1$ further rows of blocks corresponding to vertices follow, and the rightmost blocks contain the cover check areas. In all blocks corresponding to a vertex u_i and an edge e_j, a fuze is running in front of the block from the lower to the upper border. At the rear of the block, a fuze is running from left to right. If u_i is incident to e_j, then a fuze is running from the fuze in the front to the fuze in the rear, containing a diode such that the front fuze will not burn even if the rear fuze is burnt. In the rightmost blocks, the rear fuze leads to the cover check area, such that the cells in that block that did not topple during $\text{Rel}(c+b)$ topple, if one of the fuzes corresponding to the vertices incident to the corresponding edge is burnt.

In order to prevent a fuze and the cover check areas from being affected by grains of sand added outside the blocks they are contained in, we will make sure that there is a sufficient safety distance from a fuze and a cover check area to the borders of the block it is contained in by making the blocks large enough.

5.3 Construction

Consider a sandpile model on an $n(4n+1)+5 \times 4n+2 \times m(4n+1)$ grid, divided into $n \times m$ blocks of size $4n+1 \times 4n+2 \times 4n+1$ (one block per vertex and edge of G) and m blocks of size $5 \times 4n+2 \times 4n+1$.

We say that the block $(*, *, z), j \cdot (4n+1) - 4n \leq z \leq j \cdot (4n+1)$ and the middle plane of this block $(*, *, j \cdot (4n+1) - 2n)$ correspond to the edge e_j, and that the line $(i(4n+1) - 2n, 1, *)$ corresponds to the vertex u_i.

All cells of the line corresponding to u_i, $1 \leq i \leq n$ contain five grains of sand. (These cells form the fuzes as described, where the fuze at $(i(4n+1) - 2n, 1, *)$ corresponds to the vertex u_i.)

			p_1					p_2								
	...		5			...		5			...					
	...		5			...		5			...					
	...		5	2		...		5	2		...					
	...		5	5		...		5	5		...					
	...		4	5		...		4	5		...					
	...		5	2		...		5	2		...					
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	...		5			...		5			...					
	...		5			...		5			...					
3	...	2	5	2		...	2	5	2		...	2	5	5		
5	...	5	5	5	5	...	5	5	5	5	...	5	5	5	0	0

Table 1. The plane corresponding to e_j before $\text{Rel}(c + b)$.

Let $1 \le j \le m$ and $e_j = \{u_{j_1}, u_{j_2}\}$, $j_1 < j_2$ be the corresponding edge of G. Let $p_k = (4n + 1) \cdot j_k - 2n$, $k \in \{1, 2\}$ (these are the x-coordinates of the lines corresponding to the vertices u_{j_1} and u_{j_2}.)

Then the plane corresponding to e_j shall have the structure shown in Table 1. (The cells containing five grains of sand in the last row represent a rear fuze as described earlier. Further, the cells $(i \cdot (4n + 1) - 2n, 2, j \cdot (4n + 1) - 2n \pm 1)$, $1 \le i \le n$, $1 \le j \le m$ shall contain two grains of sand.)

All cells whose value is not given contain three grains of sand. This is to make sure that each cell not containing five grains of sand in c_G contains at most four grains of sand after each cell containing five grains of sand has toppled once.

During $\text{Rel}(c+b)$, all cells except those originally containing no grain of sand topple (this can be easily shown by induction). These cells form the cover check areas for the edges of G. Each cell, except the cells z with $c_G(z) = 0$ and those which are neighbors to these cells, lose as many grains of sand as they gain from their neighbors or by adding b. Therefore, the plane $(*, *, j \cdot (4n + 1) - 2n)$ has in $(c + b)_{\text{rel}}$ the structure shown in Table 2 (note the change at the lower right corner).

5.4 dist(c_G) ≤ minVC(G)

Consider a minimum vertex cover $S = \{u_{i_1}, \ldots, u_{i_k}\}$ of G. Then add a grain of sand to the lowest cell of each line corresponding to a vertex in S.

Since all these cells contain five grains of sand, they will topple and lose one grain to the cells directly above them. These also contain five grains of sand, and so all cells along the lines corresponding to vertices in S will topple.

		p_1				p_2								
	...	5			...	5			...					
	...	5			...	5			...					
	...	5	2		...	5	2		...					
	...	5	5		...	5	5		...					
	...	4	5		...	4	5		...					
	...	5	2		...	5	2		...					
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	...	5			...	5			...					
	...	5			...	5			...					
	...	5			...	5			...		5	5	2	2
5	...	5	5	5	...	5	5	5	...	5	5	4	5	5

Table 2. The plane corresponding to e_j after $\mathrm{Rel}(c + b)$.

Looking at a plane whose corresponding edge contains the vertex u_{i_j} for some j, it is easy to verify that the two cells of the cover check area of this edge will topple.

Since each edge contains at least one of the vertices $u_{i_1}, \ldots, u_{i_k}$, all cells that did not topple during $\mathrm{Rel}(c + b)$ will topple during the relaxation of the new configuration.

Thus, the configuration we get by adding the grains of sand as described to c_G is recurrent, and $\mathrm{dist}(c_G) < k$.

5.5 Adding grains of sand outside of allowed regions

Consider the blocks P_j corresponding to the edges $e_j \in E$ and the tubes T_i which contain all cells having at most the distance $2n$ from the line corresponding to u_i. So, the sets T_i form disjoint tubes that contain the fuzes.

Let $e_j \in E$ be an edge containing the vertices u_{j_1} and $u_{j_2}, j_1 < j_2$. Let z_j be a cell of the cover check area of e_j.

Let c'_G be the configuration we get by adding a grain of sand to each fuze corresponding to a vertex $u \notin e_j$ and relaxing the resulting configuration. If we now add at most n grains of sand to the configuration and the cell z_j topples during the relaxation, then at least one grain of sand has been added to the set $B = (P_j \setminus \cup_{i \notin \{j_1, j_2\}} T_i) \cup T_{j_1} \cup T_{j_2}$ (i.e., at least one grain of sand has been added to a cell "nearby" either the fuzes corresponding to the incident vertices or the plane corresponding to e_j).

Proof. Assume that this is not the case. Then we can add n grains of sand such that each grain of sand is added to a cell farther away than $2n$ from any of the following structures:

- the fuzes corresponding to u_{j_1} and u_{j_2}
- the rear fuze of the plane corresponding to e_j

- the connections between these fuzes
- the cover check area of e_j

and z_j topples during the relaxation.

First of all, it is easy to check that the fuzes incident to e_j do not burn after having added one grain of sand to each fuze not incident to e_j by looking at all possible planes. Note that for every fuze incident to e_j there is a "diode" in every plane corresponding to an edge incident to this fuze such that no grain of sand can fall on the fuze if another fuze burns. Therefore, the cell z_j does not topple during the relaxation of the configuration we get by adding one grain of sand to each fuze not incident to e_j.

It is easy to see that, since all fuzes not corresponding to u_{j_1} or u_{j_2} have been burnt, the only cells containing five grains of sand lie in the rear fuze and cover check area of the plane corresponding to e_j, in the fuzes corresponding to u_{j_1} and u_{j_2}, and in the fuzes running from these fuzes to the rear fuze.

So, if we add n grains of sand to cells farther away than $2n$ from the rear fuze or cover check area in the plane corresponding to e_j or the fuzes corresponding to u_{j_1} respectively u_{j_2} no cells in the cover check area or the fuzes can topple according to Lemma 3. Therefore, z_j cannot topple, which contradicts the assumption. ■

5.6 dist(c_G)=minVC(G)

We already have shown that $\mathrm{dist}(c_G) \leq \mathrm{minVC}(G)$. Now, assume $\mathrm{dist}(c_G) < \mathrm{minVC}(G)$ (and therefore also $\mathrm{dist}(c_G) \leq n$), and let c' be the configuration we get after adding $\mathrm{dist}(c_G)$ grains of sand, such that c'_{rel} is recurrent.

Assume that there exists an edge e, such that no grain of sand has been added to either the block containing the corresponding plane (minus the tubes surrounding vertices not incident to e) or the tube surrounding the fuzes incident to e (as in the previous proof). Then add a grain of sand to all fuzes in $(c_G + b)_{\mathrm{rel}}$ not incident to e and let them burn.

As shown earlier, even if these grains of sand are added, z_j cannot topple if less than n grains of sand are added afterwards. Therefore the assumption that the resulting configuration is recurrent was wrong since z_j has not toppled during $\mathrm{Rel}(c_G + b)$, and at least one grain of sand must have been added either to the block surrounding the plane corresponding to e (minus the blocks containing nonincident fuzes) or one of the blocks containing the incident fuzes.

Consider the following construction of a set S of vertices: For each edge e, check whether a grain of sand has been added to one of the blocks containing a fuze incident to e. If so, add the corresponding vertex (or vertices, if a grain of sand has been added to both blocks) to S; else, a grain of sand must have been added to the block containing the plane but not to a tube containing another vertex. In this case, choose one of the vertices at random and add it to S.

Since no two tubes containing fuzes have common cells, we get a set of at most $\text{dist}(c_G)$ vertices. Further, S is a vertex cover, since for each edge one of the incident vertices is contained in S. This is a contradiction to the assumption that $\text{dist}(c_G) < \text{minVC}(G)$, and it follows $\text{dist}(c_G) = \text{minVC}(G)$.

Since the construction of the configuration only takes polynomial time and the number of cells is polynomial in n, it follows that DIST is NP-hard.

5.7 DIST ∈ NP

The maximum number of topplings that can occur when one grain of sand is added to a stable configuration with n cells is bounded by n^2, as can be shown with an argument similar to the proof of Lemma 2. As the cell space is a block and each cell is less than n cells away from a border of the block, no cell can topple more than n times, and we get at most n^2 topplings in all. Therefore, if we add at most $5n$ grains of sand there are at most $5n^3$ topplings. Since the configuration d satisfying $\forall z \in Z : d(z) = 5$ is recurrent, we know that for all configurations c $\text{dist}(c) \leq 5n$.

Since it takes only time polynomial in n for a nondeterministic Turing machine to check whether or not k grains of sand are sufficient to get a recurrent configuration if $k \leq 5n$, a nondeterministic Turing machine finds $\text{dist}(c)$ in a time polynomial in n. Therefore, DIST ∈ NP meaning that DIST is NP-complete.

6. Future work

If a one-dimensional sandpile model is considered, the DIST problem can be solved in linear time. For the dimension three (and therefore higher dimensions also), we have shown that DIST is NP-complete. The open question is how hard DIST is in the case of a two-dimensional sandpile model.

While the problem VERTEX COVER is NP-complete even if confined to planar graphs, the construction for the three-dimensional case cannot be used for the two-dimensional case. One problem is the fact that a set of fuzes that form a closed curve around a recurrent sub-configuration induce all cells inside the curve to topple if the fuzes are burnt. This problem does not exist in three or more dimensions.

However, we are considering another construction using VERTEX COVER for planar cubic graphs and are confident that we will be able to show that DIST is NP-complete in the two-dimensional case.

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