The "Something From Nothing" Insertion Point: Where NKS Research into Physics Foundations Can Expect the "Most Bang for the Buck"

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1: Introducing the "Box-Kite Papers" and Zero-Divisors Generally

The focus of this presentation is the largely unexplored subject of "zero-divisors," (henceforth, "ZD's") which are forms of (hypercomplex) imaginary numbers with the peculiar property that two such, if chosen properly, are both substantial quantities – yet they have, as their product, *nothing*. Like the gnarly "coastline curves" classical analysis dismissed as "monstrosities" until Mandelbrot tamed them, then generalized "fractals" from the examples they offered, such algebraic beasts have been deemed "pathological" ever since a celebrated proof of Hurwitz's showed, a little over a century ago, that such entities are not just oddballs, but unavoidable.[1] (And, a little less than seven years ago, the Mexican mathematician Guillermo Moreno[2] showed more: in the 16-D imaginaries called "Sedenions" where we're first forced to deal with them, ZD's conform to some remarkably familiar algebraic patterns which have echoes in all higher, 2^N-dimensional, extensions of imaginaries. This led immediately to my own research, and a series of three monographs and counting which I call, for visually intuitive reasons I'll get to in short order, the "box-kite papers," shorthanded herein as BK1, BK2, and BK3 respectively.[3])

What Hurwitz showed in 1896 marked the end, in one key sense, of the revolution begun in the 1810's by taking those entities Descartes had derisively deemed "imaginary" seriously. The stupendously deep rethinking of all mathematics they facilitated in the hands of Gauss, Galois, Fourier, Reimann, et al., led some people to suspect a general strategy: after many years' frustration, William Rowan Hamilton was able to generalize the 2-D algebraic geometry of the complex plane to the 4-D context which the science of mechanics, as he envisioned it, required: in a fit of inspiration, he carved the famous formula "$i^2 = j^2 = k^2 = ijk = -1$" with a penknife on the bridge he was crossing. What made finding this easy equation so difficult is this codicil: ij = -ji, and ditto for jk (and -kj) and ki (and -ik), which is to say Quaternion multiplication is *anti-commutative*.

Those familiar with the Feynman lectures will know, though, that this is precisely what's required for a basis of vectors describing the way real forces work (which the great Brooklyn physicist was wont to demonstrate by rotating books and other objects in two different senses, in differing sequence, in class). If losing the commutative law of multiplication is the price you pay, then, for the calculus of dot and cross products (without which Maxwell's equations, among other things, would hardly be thinkable), why complain? And some contemporaries of Hamilton not only didn't grumble, but they sought to push the envelope even further, which led Cayley and Dickson and Graves to double dimension once again, yielding the 8-D Octonions, wherein even the *associative* law breaks down. This means, among other things, that their symmetries can't even be codified with group theory, since simple expressions like "abc" become suddenly problematic: for, generally speaking, $a\cdot(b\cdot c)$ and $(a\cdot b)\cdot c$ may well have opposite signs! (Hence, algebraists concerned with such things speak of "loops" not groups, which do for nonassociative operations what groups *ought* to do but can't![4])

Algebraically, the number fields of Imaginaries, Quaternions, and Octonions (usually indicated by **C** for *complex*, **H** for *Hamilton*, and **O**) all conform to a "squares rule": expressing, say, an arbitrary complex number as $\mathbf{z} = \mathbf{a} + \mathbf{b}\boldsymbol{i}$, and another as $\mathbf{z'} = \mathbf{c} + \mathbf{d}\boldsymbol{i}$, we take for granted that the absolute value of their product is equal to the product of their absolute values (i.e., it "has a norm"); alternatively, writing things out in terms of components, we get a relationship between squares (i.e., "the product of *one* sum of two squares by *another* is a sum of two squares as well"): $\mathbf{(ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2)}$

This generalizes to sums of 4 and 8 squares, for Quaternions and Octonions in that order, and the formulas, while excruciatingly tedious to calculate in the pre-computer days when Cayley and company were investigating all this, are conceptually, at least, rather easy to grasp. People naturally wanted to keep doubling dimensions, to find even weirder and niftier number fauna, but Hurwitz spoiled their fun by showing that one more doubling to 16-D induces the loss of something even more critical than associativity: the very notion of a norm (and hence possibility of a straightforward "squares rule"[5]) goes out the window. That is, defining a metric (and hence, standard "field" structure) becomes problematic; corollarily, you get zero-divisors (i.e., number-theoretic "black holes").

That last parenthetical remark, though, suggests a context wherein Sedenions, and even higher 2^N-ions, might prove of real use: studying *quantum gravity.* Of the many aspects of the latter that make it hard to integrate with standard QM, one in particular makes the breakdown of norms spring to mind: the index of what we might call an electron's "self-absorption," the "fine structure constant" of 1/137, has analogues for the weak and strong forces, but for gravity the "constant" in fact is a *variable*! A survey by three Estonian physicists of <u>Nonassociative Algebras in Physics</u> frames this in terms of "infobarriers," or "certain limits in obtaining information" – related to the degree of "hypercomplexity"required to study their limit-case phenomena (linked in turn to certain limit-case constants, such as Einstein's *c* and Planck's *h,* for **C** and **H** respectively):

> Complex numbers seem to have a fundamental connection with the special theory of relativity allowing to introduce the light cone and a nondetermined interval for events. The noncommutativity of QM is related to the uncertainty relations and the complementarity principle, and is characteristically represented by the quaternion units satisfying the same commutation relations as do the angular momentum operators.[6]

The authors also discuss Octonionic arguments advanced since the 1970's to model a third infobarrier, associated with the strong force: *quark confinement*. And clearly, Octonionic modeling has been running rampant since the advent of superstrings and M-Theory, where a stress that sometimes verges on "negative theology" is placed upon the implications of certain kinds of *non-observability*.[7] But more germane here is a brief "blue sky" piece by Andrei Sakharov[8] concerning an underlying spacetime "lattice" structure (the lattice constant being the Planck length) subtending gravity, linked to a spontaneous, post-Big Bang, phase transition – with gravity as some kind of *elasticity* of this lattice (and hence of its "metric"). Our Estonian authors see a role here in studying this fourth infobarrier involving the Sedenions – at which point, their book ends, and my own thoughts (with progressively emerging links to NKS modeling) begin . . .

The Estonians play with various preparations of Sedenions which serve to mute or suppress their distinctive zero-divisor features: "binary" and "ternary" Sedenion schemes which can be almost tamed by matrix methods, etc. This is understandable, as they were thinking about "infobarriers" at a time when ZD's still seemed "pathological." But the history of the hypercomplex should suggest to us that even the loss of structure implicit in ZD's portends some unsuspected gain in other aspects. Just what might this be? The answer is quite surprising: contrary to a century's intuition in this area, what I find is a tendency toward spontaneous cobbling together of ever more elaborate structure as the N in 2^N-ions gets bigger; more, that this "mustering" (and its reverse process, which I think of – to continue the military analogy – as "demobilizing") is necessarily interlinked with the survival of "classical" structures (specifically, associative triplets, clones of the Quaternions' "i, j, k" , which interpenetrate in a manner at once remarkably simple, yet exceedingly rich from a modeler's vantage), in cloistered algebraic "safe-houses" whose building blocks are the octahedral lattices of ZD's I call "box-kites."

From here on out, I will have much less to say about physics per se, and much more about mustering as a general NKS phenomenon. Indeed, the title of this presentation was submitted while I was still framing BK3 – but, much to my surprise, the physics focus of the results presented therein took shape much more quickly than anticipated, leading me to feel far less interested in regurgitating results just written up. So I only note briefly two key themes. First, the algebraic "homomorphism" Moreno found was to the automorphism group of the Octonions, known in Dynkin-diagram argot as "G_2"; but this is the same 7-fold-symmetric patterning which Dominic Joyce has used to model the 7 "curled up" dimensions of "gauge forces" in M-Theory – and this "Joyce manifold" is built upon 35 equations. (Also the number of associative triplets in the Sedenions: the Octonions have 7; the Quaternions, but 1; and 2^N-ions in general have $(2^N\text{-}1)(2^N\text{-}2)/(3!)$.) Second, the breakdown of the "squares law" beyond the Octonions opens out on a way of thinking surprisingly akin to renormalization: if the product of a pair of 2^N-ions seems to need the number of dimensions in which 2^{N+1}-ions live, and if "look-alike" algebras incorporate "sleeper-cell structures" (i.e., dormant ZD's whose ZD-ness is unobservable as such), with the same multiplication tables as Quaternions and Octonions, but with their probabilistic nature *implicit in* (not *superimposed upon*) their workings . . . if such ideas are of interest to you, please go to BK3. For I'll have no more to say about them here.

What I *do* want from physics, to start with, is what one philosopher of science has called a "remote analogy"[9]: one which, if not taken too literally, can help guide our thoughts into the strange terrain we're entering. I refer to the cosmological notion which has only been taken seriously for about 4 years now, known as "co-evolution." Since John Kormandy's presentation at the June, 2000, AAS meeting, it has suddenly become a mainstream notion to see supermassive black holes as "galactic sculptors" – that is, necessary ingredients in galactic evolution, at least for elliptic galaxies like ours, with a clear correlation between the size of the central black hole and the galaxy itself.[10] I invite you to jettison the "common wisdom" of ZD's as "pathological," and imagine an indefinitely large (in theory, up to infinite-dimensional) space of ZD's – replete with a vast potential for "mustering" and "demobilizing" structures, be they evanescent or long-term, guided by a "dark matter halo" of NKS protocols still to be determined – as the "supermassive black holes" which center and structure the galaxies of *number*.

No

2: The Theory of ~~Every~~thing: Starter-Kit Concepts

BK3 ends with two pages presenting and generalizing from a “synchronization table” displaying a fundamental property I call “trip-sync,” for short for **sync**hronization of associative **trip**lets (a.k.a., systems of mutually interpenetrating Quaternion copies). Other phenomena (in particular, the “sand mandalas” in the 32-D “Pathions,” whose graphical sequencing in BK2 – reproduced below – was the first clue to NKS behaviors) *suggested* complexity in the NKS sense. But the “spin control” implicit in the “slippage” between triplets sharing Box-Kite pathways shows us where it's hiding and *what it feeds on*. As happens surprisingly often in mathematics, being an inch away from a Big Idea can feel like a mile until you get there, so while the closing thought in BK3 was that it was an open question whether the Trip-Sync property holds for ZD patterns in *all* 2^n-ions, n > 3, it turned out to be embarrassingly easy to prove it in full generality, using nothing much more than the defining properties of Box-Kite lattices themselves. So Trip-Sync's deployment will mark the end-point of the presentation here of the “starter-kit concepts” – after which, the “work in progress” of “mustering theory” proper can be broached.

To get to a point where “trip-sync” will seem a natural thing to contemplate, we need to consider, first, notation; next, the Cayley-Dixon Process (or CDP), the algorithm which allows us to build 2^{N+1}-ions from 2^N-ions for N arbitrarily large; then, Box-Kites themselves, which the right notation will make easy to explain, and ensembles of which, in higher-than-16 dimensions, CDP machinery will let us investigate and navigate with minimal pain.

While the Quaternions have but two distinct representations, hence two labeling schemes (ij = k, or ik = j), the Octonions have no fewer than 480, and the Sedenions a few hundred billion. Nor are all the Octonionic schemes simply “equivalent” in all situations; nor are arbitrary labeling schemes for the Sedenions necessarily useful or even feasible for our purposes. (For details, see BK1.) Some culling is in order, and a guiding principle to motivate it. It is found in an outrageously simple-minded convention – highly suggestive, in formal outline, of the definition of “norm” governing number forms less “hyper” than Sedenions: for the 2^N units in a given universe of 2^N-ions, label their axes with subscripts running from 0 (for the reals) to 2^N-1, so that products of two such units will have indices equal to the XOR of the indices of their factors. This is not only do-able for all N, but simple to work with to boot.

For the Quaternions, the rule is trivial: instead of i, j, k, write i_1 , i_2 , i_3; and, since 1 *xor* 2 = 3, the product of any pair of axes yields the third. What's not so trivial is the *sign*, though, of the resultant unit. And since, with the Octonions, multiple labeling schemes conforming to our XOR rule are possible (and actually in use), we choose that one which results from the most straightforward deployment of CDP. For more than you probably care to know about this algorithm, see the background discussion and “bit-twiddling” proofs in BK2. For now, I'll just provide a standard, readily generalized, way of writing Octonion triplets, then expand my interest to embrace the Sedenion multiplication table; the iterative procedure one can point to in moving from the one to the other will suggest a very simple set of rules (even dumber than those I reduced things to in BK2) for *any* 2^N-ion tabulations, which I'll describe in ready-to-implement pseudocode fashion.

As composing strings of subscripted "*i*"'s is quite tedious, let's just write the indices in comma-delimited, parenthesized lists of 3, in cyclical order. Put the lowest index number first, then the index whose product with it yields the *positive* value of the third. The Quaternions' "i, j, k" hence gets signified this way: (1, 2, 3). The seven associative triplets (henceforth, "O-trips") found in the Octonions are written thus:

(1, 2, 3); (1, 4, 5); **(1, 7, 6)**; (2, 4, 6); (2, 5, 7); (3, 4, 7); **(3, 6, 5)**

Note two things: first, each such triplet is clearly "XOR compliant" per my earlier remarks – scanning the list right to left, 3 *xor* 6 (= 011 *xor* 110 = 101) = 5; 3 *xor* 4 (= 011 *xor* 100 = 111) = 7; ... Second, this particular mode of XOR compliance displays 2 triplets (bolded in red) where counting and cyclical orders differ: (1, 7, 6) and (3, 6, 5) yield the wrong signs as we cycle through their products if we write them (1, 6, 7) and (3, 5, 6). Labelings are possible (and actually in use among some physicists) where only *one* triplet is written "out of order"; it is not possible, however, to come up with a labeling where counting order and product signing are in accord in all 7 instances.

The Sedenions include the 7 O-trips, but also contain 28 other associative triplets (henceforth, "S-trips") which I now list (again red-bolding those "out of order"):

(1, 8, 9)	**(1, 11, 10)**	**(1, 13, 12)**	(1, 14, 15)
(2, 8, 10)	(2, 9, 11)	**(2, 14, 12)**	**(2, 15, 13)**
(3, 8, 11)	**(3, 10, 9)**	**(3, 15, 12)**	(3, 13, 14)
(4, 8, 12)	(4, 9, 13)	(4, 10, 14)	(4, 11, 15)
(5, 8, 13)	**(5, 12, 9)**	(5, 10, 15)	**(5, 14, 11)**
(6, 8, 14)	**(6, 15, 9)**	**(6, 12, 10)**	(6, 11, 13)
(7, 8, 15)	(7, 9, 14)	**(7, 13, 10)**	**(7, 12, 11)**

For the first column only, all triplets display both counting *and* signing order, so that the *xor* of the smaller pair of indices is identical to their *sum*. This is directly related to the workings of CDP: for the key is the middle term in each, namely $8 = 2^3$. The unit corresponding to this index is the CDP *generator*: in general, to get 2^{N+1}-ions from their predecessor 2^N-ions, you affix a unit of index 2^N and produce 2^N-1 triplets as above. All such generator-containing triplets are, from the vantage of ZD creation, *sterile*. Indeed, it is this sterility which leads to the very simple production rule for creating arbitrary zero-divisors: take any arbitrary Octonion of index *o* (7 choices); then take any Sedenion of index *S*, such that *S* is neither 8 nor the *xor* of *o* with 8 (6 choices); the diagonal lines in the (*o*, *S*) plane (one of the 7·6 = "42 Assessors") will have all their points be ZD's.

The tricky part is figuring out which diagonals zero-divide with which. This is quite different from the situation found in standard QM, where a special (and, from our purview, *degenerate*) case of ZD's has a crucial role to play. If we de-reify the usual operator formalism to reveal the "inner number theory" trying to get out, we can see that the "projection operators" underwriting spin-up vs. spin-down options, for instance, are idempotents which just happen to mutually annihilate each other when multiplied (i.e., they're ZD's with respect to each other). They define two diagonal lines, with each point of one zero-dividing each point of the other. Now let's say this again, in simpler English.

We know that a Complex "line" is really a *plane*; the product of two such lines, then, is really a 4-space. This space is *not* navigated by the Quaternions, since it's produced by 2 real axes and 2 imaginaries; *but:* in one of the great magic acts of modern mathematics, this "bicomplex" 4-space (as navigated by the "unitary rotation group" "SU2") is *isomorphic to* the Quaternions. The "bicomplex" space is also the simplest interesting case of the general associative vector space called a Clifford algebra (the one dubbed "Cl_2," since it has 2^2 dimensions, to be precise). Its axes include the usual real and imaginary; but the other two are "mirror numbers" (they square to +1, not – 1), and a form of *i* which operates commutatively on the other axes' units. This last is represented by 2 x 2 matrices with the usual *i* replacing the 1's in the identity matrix. And since Wolfgang Pauli's day, the "mirror numbers" (one kind corresponding to each Quaternion unit) have been represented by the trio of 2 x 2 "Pauli spin matrices." (All 3 of these, plus the Quaternions, plus the "commutative imaginaries," together span the 8-D space called "Cl_3"; we won't be needing to know about this here, though, fortunately!)

If we signify "mirror numbers" by the letter *m*, then the standard projection operators can be written thus: $\frac{1}{2}(1 \pm m)$. It's easy to show that, for each of these quantities, raising it to any arbitrary power leaves it unchanged; hence, each operates along its diagonal in the (1, *m*) plane like the '1' on the real axis – which is to say, it's *idempotent*. It's also easy to see that their product (and hence that of any point on the one diagonal times any on the other) displays *nilpotency* as well, since $(1 + m)(1 – m) = 1 + m – m – 1 = 0$.

The diagonals of the 42 planes I've called "Assessors" in the Sedenions are likewise ZD's, but with a difference: they *never* mutually zero-divide, and they *only* zero-divide with points on one or the other (but *never both*) diagonals in some other Assessors. And we can narrow this down further: they only zero-divide with *four* other Assessors, in such a manner that points on the "\" diagonal in one will zero-divide with points on *one* of another Assessor's diagonals, while the "/" diagonal will zero-divide with the *other*.

More specific yet: unlike "projection operators," which always come in pairs, Assessor planes containing mutually zero-dividing diagonals come in *threes*. And, since each Assessor participates in two such (otherwise unconnected) trios, it turns out that the simplest layout is actually the one followed: i.e., the ZD activity of each Assessor is constrained within an octahedral arrangement of four "Co-Assessor trios," where each vertex represents an Assessor plane, and the trios only share vertices, not edges, with each other.

Note, too, the "edge signs," indicating which diagonals form ZD's with which, when two Assessors joined by an edge are considered: if diagonals must have the *same* orientation (" \ · \ " or " / · / "), mark the edge "+"; otherwise, write "– ". Then color in the 4 triangles whose edges form "closed cycles" (for vertices A, B, C, each Assessor pairing "emanates" the third: e.g., $(S_A, o_A) \cdot (S_B, o_B) = + (S_C, o_C) – (S_C, o_C) = 0$) and call them "Sails." Call the other 4 "Vents," and you get an octahedral "Box-Kite" which can fly in Sedenion winds. 7 such, all isomorphic, partition the 42 Assessors, each taking 6; from these simple structures, the entire "pathology" of Sedenion ZD's can be grasped.

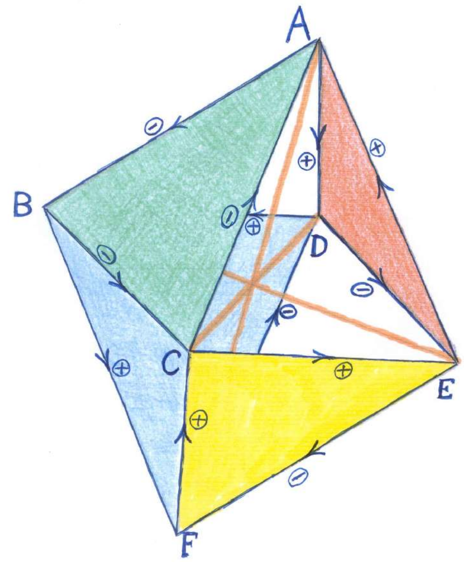
The monicker "Box-Kite" is further justified by the fact that while the real-world variety have wooden struts separating opposite vertices, thereby creating support and stability sufficient to let them fly, "struts" provide the crucial ingredient in our ethereal Box-Kites as well: the 3 pairs of Assessors joined by "struts" do *not* mutually zero-divide!

SEDENION "STRUT TABLE"

Read off struts in "nested parentheses" fashion:

AF, BE, CD = the 3 Struts
ABC, ADE, FCE, FDB = the 4 Sails
(with ABC the "Zigzag" Sail)

Box-Kite	Goto[11] #'s	A	B	C	D	E	F
I	7,6,4,5	**3**, 10	**6**, 15	**5**, 12	4, 13	7, 14	2, 11
II	3,2,6,7	**1**, 11	**7**, 13	**6**, 12	4, 14	5, 15	3, 9
III	5,4,2,3	**2**, 9	**5**, 14	**7**, 12	4, 15	6, 13	1, 10
IV	1,3,5,7	**1**, 13	**2**, 14	**3**, 15	7, 11	6, 10	5, 9
V	4,1,6,3	**2**, 15	**4**, 9	**6**, 11	3, 14	1, 12	7, 10
VI	6,1,2,5	**3**, 13	**4**, 10	**7**, 9	1, 15	2, 12	5, 11
VII	2,1,4,7	**1**, 14	**4**, 11	**5**, 10	2, 13	3, 12	6, 9



Struts have another major property: products of all 3 "strut pairs" in a Box-Kite obey the same pattern; the (*o, S*) multiplications always yield the *generator* (the index-8 unit in the Sedenions); the (*o, o*) and (*S, S*) pairings, though, always result in the "missing Octonion" (since an Octonion index resides at each vertex, of which there are 6, one *must* be absent!) which serves as the signature, or "strut constant," of the Box-Kite in question. To differentiate it from an index proper, I write it with a Roman numeral, as shown in the left-most column in the "Strut Table" above.

The 12 "edge-signs," drawn above right on the Box-Kite, evenly divide between "+" and "– "; but "– "'s cluster in such a manner as to form the edges of two opposite faces of the octahedron, one of which is a Vent, the other a Sail. The convention here is to assume this Sail is ABC (making its corresponding Vent DEF). This special Sail I call the "Zigzag," since traversing the edges so that one always "makes zero" creates a 6-cycle of flipflopped traversals: multiply mutual ZD's from A and B, then B and C, then C and A, and begin, say, with the "\"; the lines picked will keep switching like this: "\ / \ / \ /".

Let's make this concrete: consider the Zigzag of Box-Kite I. Start with the "/" diagonal in A's plane (which the table tells us is spanned by i_3 and i_{10}), and proceed as just indicated. Here's the actual 6-cycle of ZD products (ignoring constants, since *any* points on the indicated lines will serve as well as any others):

$$(i_3 + i_{10})(i_6 - i_{15}) = (i_6 - i_{15})(i_5 + i_{12}) = (i_5 + i_{12})(i_3 - i_{10}) =$$

$$(i_3 - i_{10})(i_6 + i_{15}) = (i_6 + i_{15})(i_5 - i_{12}) = (i_5 - i_{12})(i_3 + i_{10}) = 0.$$

The other 3 Sails (ADE, FCE and FDB) all have 2 "+"'s and just one "– " for edge-signs. Their 6-cycle patterns – "\ \ \ / / /" or "/ / / \ \ \" – suggest the knots they're named for: Trefoils. This Zigzag/Trefoil difference is the basis of the Trip-Sync property; all we need do now is consider the O-trips and S-trips which Sails are made of, and "blow up" each Sail into a kindred "Box-Kite" layout. Only on *these*, "Sails" now represent the 4 distinct *triplets* weaving through a Co-Assessor trio, with the sole O-trip cast in the "Zigzag" role, and the 3 (*o*, *S*) pairs which span Assessor planes bounding the "struts."

This is much easier to follow by seeing it than hearing it described. So let's take as for-instance the Zigzag of Box-Kite I. Its 3 Assessors' (*o,* S) pairs have the indices given in the prior page's "Strut Table": (3, 10); (6,15); and (5, 12). The 3 *o*'s comprise one of the standard O-trips; but the "recursive Box-Kite" described last paragraph (or "Rbox" for short) shows that each of these 3 *o*'s forms an S-trip with the other two trio members' Sedenions. Hence (**3**, 15, 12); (10, **6**, 12); (10, 15, **5**) – written in A, B, C order with the Octonions in bold – are all S-trips we can find in the listings given above.

Two differences should be noted between a Sedenion Box-Kite ("Sbox") and the Rboxes one can build on its Sails. First, "double covering": the Sbox "Sails" are 6-cycles, those of an Rbox are 3-cycles. Second, "mutual mirroring": the orientation of the cycling (as indicated by the arrows inscribed along the diagram's edges) is reversed when one moves between Rbox and Sbox representations. This is a side-effect of two things which are ultimately the *same* thing: "bit-twiddling" (pure Sedenions must be written with an extra bit when we XOR them, so that the XOR of two *S*'s is always an *o*); and, diagonal skewing (the Sbox places *S*-terms, which flow with the *o*-term's cycle in the Sbox, on the other end of the strut the *o* and *S* terms span in the Rbox).

The "mutual mirroring," then, is an artefact of the Rbox representation. But it serves to indicate the nature of the property we're after: for orientation-reversal is what Trip-Sync is all about. Reverting to the Sbox we started with, the *o*'s at each vertex of a Sail belong to 2 different triplets; in the Zigzag's case, irregardless of the added "weight" the S-trips carry (an extra bit to the left when we write their indices in binary), the orientation is unchanged as we move from O-trip to S-trip. Hence, we could imagine a sort of "slippage" as we define Quaternionic orbits deformable into the simple i,j,k cycling round ABC, wherein the extra bit is "added" or "dropped" without our being cognizant of that fact. If the two orientations be thought akin to QM-like "spin up" vs. "spin down," we could even claim there was an "infobarrier" forbidding us to detect which case obtained.

We cannot say the same, though, in the general case: for, with Trefoils, S-trip to O-trip "slippage" causes a *reversal* of orientation, whenever the *o* shared between them is *not* one of the Zigzag's trio (i.e., 2 times out of 3). Consider, for instance, the ADE Sail in Box-Kite I. Writing the S-trips per above (with O-trip components in bold), we get this: (**3**, 13, 14); (10, **4**, 14); (10, 13, **7**), with each triplet in A, D, E order. A quick look at the earlier listing indicates that the second and third of these are written in backwards order, whereas the first (which shares the **3** with ABC) can be "let slip" without our noticing it. This is the ground floor, then, of observability (and, as such, bears a deep relationship to Penrose's "spin networks: a topic for another time). What is more, it holds true for *all* Box-Kites, in *all* 2^N-ions. *This* is the Trip-Sync property. *Proving* it is easy.

Let's write any *S* index using uppercase, and any *o* in lowercase; let's further write "*sg*" to designate a binary variable equal to ± 1. Then any Assessors (K, k) and (L, l), if joined by an edge-sign marked "–", will have product $(K + sg\cdot k) \cdot (L - sg\cdot l) = (KL - sg\cdot Kl + sg\cdot kL - kl) = 0$. Since KL and kl = *o*'s, and Kl and kL = *S*'s, then KL = kl = m, and Kl = kL = M, where (M, m) is the third Assessor in the Sail. If the edge-sign be "+," then the opposite obtains: KL = –kl = –m, so S-trip (K, L, m) has opposite sense from O-trip (k, l, m). Since the edge-signs leading *out* from a Zigzag vertex in a Trefoil must be "+," the edge *opposite* said vertex in the Trefoil will be marked "–"; hence only the lone S-trip in this Trefoil which includes a Zigzag *o* will "let things slide." (QED – no kidding!)

This may not seem like such a big deal: at least, not within the Sedenions. But as early as the 32-D Pathions, larger ensembles present themselves – seven Box-Kite septets I call "Pléiades" emerge, which interconnect 14 Assessors, each of which belongs to 3 Box-Kites and 6 Sails. Sharing these properties but not others is the "middleman" between them and the "Sand Mandalas," which I call the "Muster Master." Each of its 7 Box-Kite Zigzags is exclusively mapped to an O-trip – which means, if we treat "spin-flips" as units of attractive or repulsive *force*, the Muster Master can round up (or chase off) any and all Sedenion Box-Kites by an extension of Trip-Sync slippage.

Given its unique holding capacity (harboring 1 copy each of every O- and S-trip appearing in any Sedenion Box-Kite), the Muster Master effectively serves, then, as a Pathion "Atlas" for all ZD patterns on the other side of the 2^4-ion/2^5-ion divide. Given, further, that the word derives from the name of the Titan in Greek myth who kept the earth and heavens separated by brute strength – and who also fathered the seven sisters called the "Pléiades" – we could even say "Atlas" is the "Muster Master"'s proper name!

If we assume, as well, that Box-Kites are free-floating, each with its own copy of (or window on) Zero, and its own clones of appropriate 2^N-ion units acting as "internal coordinates," wild "population dynamics," with all sorts of "clumpings" and "schisms" in evidence, suggest themselves. Further, given the simplicity of the latticework connecting Assessors and their composites, and the binary nature of the "spin control" we can put in play among them, the simplest approach to their study seems to mandate NKS setups. The final pages will provide broad and sharp-edged guidelines for pursuing them.

The remainder of this presentation will be focused on three tasks. First, if we must (rather literally) "redouble" our efforts by moving from Sedenions to higher reaches (to the 32-D Pathions at least, where "Sand Mandalas," "Pléiades," and the "Muster Master" live), we'll need to redeem an earlier promissory note, and provide some "pseudo-code"-level detail concerning general 2^N-ion multiplication. Next, we'll probe the phenomena just alluded to in 32-D, wherein NKS patterns respectively were first suspected, and NKS tactics seem most clearly motivated. Then we'll hone in on a surprisingly deep connection between ZD's and Boolean logic, which I'm only beginning to grasp myself – which will lead me to rough-sketch a promising possibility for fusion of methods.

At April's conference, my own poster was situated a few feet away from Rodrigo Obando's in the presentation gallery, and what I gathered of his work from discussions there and later surprised and excited me. Certain "magic numbers" were common to both our approaches, for reasons neither of us clearly understands yet. The order of the second "simple group," 168, plays a critical role in his Boolean-function approach toward generating (r = 2) NKS complexity from logical "first principles" (it's the number of both iso- and anti- tone functions with 4 variables, a result which goes all the way back to Dedekind). In my own eliciting of ZD "head counts," 42 Assessors implies168 oriented lines of ZD's, as well as the same number of "edge currents" (2 per edge, 12 edges per Box-Kite), both echoing the order of PSL(2,7), the finite projective triangle which houses Octonion labeling schemes – and pops up again when investigating fixed "strut-constant" slices of the Pathion pie . . . about which, I can't say anything further until I've deployed the arguments just promised. But I *can* say that such surprising "hybrid fusion," between such disparate points of view, is what conferences like NKS 2004 are meant to facilitate. What better theme, then, could present itself for bringing these pages to a close?

3: Revisiting the CDP Algorithm; Invading the 32-D of the Pathions

Let's return to the scene of the crime: the triplet listings on page 5. Recall that the first of the 4 columns of S-trips has all its length-3 lists in counting *and* signing order (because the '8' in the middle is the *generator*). What to make of the apparently random way disordered lists display in the columns to the right? Reading from the left, there are 4, 5, and 3 in successive columns: half the 28 "8-less" O- and S- trips. Where is the pattern in this? The answer is found in the relation of the rows (one per *o*) to the 3 O-trips implicit, via "slippage," in the *o*-harboring S-trips listed across each row's "8-less" columns.

If you left-multiply i_8 by any i_o, you get + $i_{(8+o)}$ – which is an *S*. Likewise, multiplying i_8 on the left by any *S* will yield a *negative o* – of index (*S* – 8). Apply this obvious logic to the triplets on the left, operating on those in the rows whose leading *o*'s form an O-trip. In the cycle (1, 2, 3), two consecutive terms have *xor* equal to the third, with positive sign; the rightmost terms in the same rows, though, while not forming a cycle as written, imply one upon *xor*ing and sign-flipping. Paralleling 2 *xor 3* = 1, we have 10 *xor* 11 = 1, but with sign *reversed*. Whence the S-trip at the top of the 1st column, (1, 11, 10). By cycling according to the same logic, (3 *xor* 1 = 11 *xor* 9 = 2) → (2, 9, 11), and (1 *xor* 2 = 9 *xor* 10 = 3) → (3, 10, 9). By proceeding similarly with (1, 4, 5), rows 1 of the 2nd column, and rows 4 and 5 of the 1st, get filled in, while working with (1, 7, 6) completes the 1st row and 1st column. Conducting the same sort of business with the next two O-trips – (2, 4, 6) and (2, 5, 7) – suffices to finish the tabulation.

This procedure can be extended to 2^N-ions, N any integer: just replace 8 with the appropriate generator, $g = 2^{N-1}$, and create a first column with $g – 1$ triplets, one per row. The number of columns will correspond to the number of *L-* (no longer just O-) *trips*, where L indicates indices *lower* than *g* (with U indicating those we'd now signify with *uppercase* lettering, the way we used *S* for Sedenion indices > 8, which indices appear in *U-trips*). For the Pathions, this means 7 columns to the right – the number of Octonion units. For 2^N-ions generally, the number of columns to the right of the *g*-containing one equals the count of imaginary units in the 2^{N-2}-ions: 0 for **H**, 1 for **O**; 3 for **S**; 7 for **P**...

This table-building strategy is not only easy to implement, but it's a lot simpler than the way the CDP algorithm is typically explained – which means, perhaps, it should be dubbed the "Fast CDP." And it also suggests a table-free method for calculating the products of any two imaginaries of arbitrary indices. It can be summarized in 3 rules:

Triple Negatives Rule: For the general product of two units whose indices are **r** and **c** (for **row** and **column**, where we read off the left- and right-hand terms to the multiplication in an imagined table of product indices, ranging from 0 to 2^{N+1} - 1), if *both* are less than the generator $\boldsymbol{g} = 2^N$, the value in cell (**r, c**) of the table = **r·c**. If *one* term > ***g*** (it doesn't matter which), the index and sign of their product is just **[–(2^N + r·c)]**. If *both* terms > ***g***, XOR-ing kills the 2^N, leaving just **[–r·c]**.

Trident Rule: Multiplying ***g*** on the left (right) by any unit whose index > ***g*** results in a unit whose index is their sum, with same (opposite) sign. Diagonal entries in quadrants of the table (the result of products where **r** = **c**) are + ***g*** below the main diagonal (whose "index 0" entries, all squares of imaginaries, evaluate to –**1**), and – ***g*** above it (but for the 1st row, which is just the trivial, all-positive, products with the real unit).

Recursive Bottoming: Successive use of these two rules resolves all products.

The reader is encouraged to pore over the Sedenion multiplication table, the better to see how it conforms to these rules, taking *g* variously as 1, 2 or 3. A version with checkerboard shading, to make two levels of "quadrants" easier to see, can be reached by clicking the "BK1" URL in the 3rd endnote and going to page 6. As a way to gain rapid entry into higher dimensions, however, the reader is invited to consider another rule, which can generate properly signed triplets in arbitrarily high dimension.

Zero-Stuffing: Insert 0's in the same position in the binary representation of a triplet, and you'll get another, similarly signed. Taking the **H**-spanning triplet (1, 2, 3) and stuffing 0's to the right, then the middle, yields the O-trips (2, 4, 6) and (1, 4, 5). This plus the sign-flipping effect of adding 1-bits to the left of, then *xor*ing, doublets – (**1**, **2**); (**2**, **3**); (**3**, **1**) giving (**3**, **6**, *5*); (**1**, **7**, *6*); (**2**, **5**, *7*) – yields up all the O-trips save (3, 4, 7).

Zero-stuffing can generate an infinite series of "harmonics" of any given triplet; and, by extension, it can give us a similarly infinite "overtone series" of Box-Kites. This fact, plus a simple generalization of the "strut constant" notion, makes generating and classifying Box-Kites in the Pathions fairly straightforward (up to a point, as I'll shortly spell out). Take the four O-trips of one of the seven Sedenion Box-Kites, and make them the L-trips for a "base-line" Pathion Box-Kite by stuffing a zero to the right of the left-most 1-bit in each Assessor's *S* term: the strut constant remains unchanged, but the *S* terms become *U* terms, with indices increased by 8.

In the Sedenions, though, each strut constant, by dint of being a "missing Octonion," implicated 3 pairs of strut-opposite Assessors (one per each O-trip containing it). But 7 S-trips contain the Octonion, and each of these must be accommodated when we set about constructing Box-Kites in the Pathions. The upshot is that each strut constant now collects 7 Box-Kites, as there are now 2^4-2 Assessors to choose from: each of these will zero-divide all but itself and its strut-opposite, for 168 ZD products – and, as the 4 Sails of a Box-Kite each mandate 6 such products, we get 168 / 24 = 7 Box-Kites in all.

Consider the 4 Pathion L-trips which are S-trips, associated with strut constant **s** = 1: (1, 8, 9); (1, 11, 10); (1, 13, 12); (1, 14, 15). We get 8 Assessors out of them by first throwing away the 1's, then taking one of the remaining terms as given, and adding a 1-bit to the left of the other: (8, 25) and (9, 24) form a pair of strut-opposites, as do (11, 26) and (10, 27); (13, 28) and (12, 29); and (14, 31) and (15, 30). We then build Zigzags by keeping one O-strut, moving it into the A, F slots if not there to start with, and picking two S-trip Assessors among the four pair just listed which form L-trips at (A, B, C) and (A, D, E). Zero-stuffing and 1-bit appending make picking and arranging fairly easy.

For instance, keeping with the ongoing example, the L-terms at the vertices for the base-line Box-Kite with **s** = 1 are (3, 6, 5, 4, 7, 2). Keep 3 and 2 at A and F: appending 8's to the left of the (B, C), then (D, E), terms reverses their order, giving us this new Box-Kite: (3, 13, 14, 15, 12, 2). Stuffing zeros to the left of the 2^1 bit, meanwhile, keeps terms in their places, but changes the L-terms at the vertices to read this way: (3, 10, 9, 8, 11, 3). Rotating the (B, E) and (C, D) L-pairs to (A, F) scrambles the ordering a bit, but the same basic principle applies: with (6, 5) at (A, F), the "L-bit trick" gives us (after some rearrange) this list: (6, 11, 13, 12, 10, 7); the second Box-Kite with (6, 5) on the A-F strut, though, requires *both* tricks to isolate an appropriate match with the right S-trip Assessors: zero-stuffing sends (5, 4) to (9, 8); but applying the "L-bit trick" *twice* is required to send (3, 2) to (15, 14), the last available S-trip Assessors.

This last example suggests a problem: with the Sedenions, **s** ranges from 1 to 7 – this last being one less than the generator, *g*. For the Pathions, *g* = 16, so there are *15* values possible for **s**. But one of these = the Sedenions' *g* (which we might call the Pathions' "subgenerator," g^{-1}), and 7 exceed it. If multiple "L-bit trick" applications are needed frequently, might we not risk "carry-bit overflow" as **s** grows? The answer is yes.

The reference four paragraphs back to a 14 x 14 multiplication table is not quite exact, since the product of mutual ZD's, strictly speaking, is zero by definition. To be informative, the table's cells show *emanations*: per the earlier definition, diagonals of Assessors which mutually act as ZD's in fact engage in "pair production" of two oppositely signed copies of the third Assessor in their Sail. The L-term index for this third Assessor is what's shown in the "emanation table," with an underbar if the edge-sign joining its producing Assessors (whose L-term indices are shown on the row and column headings) is "–". Such a table has blank diagonals (since Assessors never make ZD's between their own diagonals, or with those of their strut-opposite), hence $14^2 - 2 \cdot 14 = 168$ filled-in cells. When **s** = g^{-1}, though, the edge-signs cluster so that all cells in the upper left and lower right quadrants are "–", while those in the opposite quadrants are all "+".

When **s** > g^{-1}, however, the ZD population suddenly collapses: only 72 cells are filled in in each, meaning 3, not 7, Box-Kites. Consider the first such case (**s** = 9):

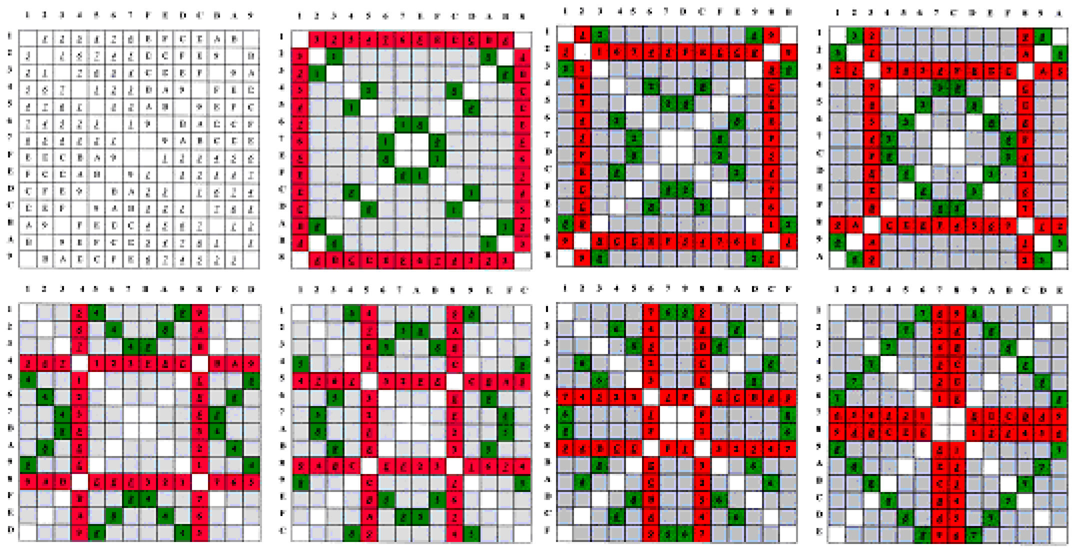
	A	***B***	***C***	***D***	***E***	***F***
Base-Line Box-Kite:	(2, 27)	(8, 17)	(10,19)	(3, 26)	(1, 24)	(11, 18)
"First Harmonic":	(4, 29)	(8, 17)	(12,21)	(5, 28)	(1, 24)	(13, 20)
"Second Harmonic":	(7, 30)	(8, 17)	(15,22)	(6, 31)	(1, 24)	(14, 23)

It is tempting to assume the 6 Assessors arrayed in the *columns* D and A, after minor rearrangement, form another in the "overtone series" of our **s** = 1 "baseline Box-Kite": but "carrybit overflow" is at work, so that left-appending 1-bits to all U-terms – converting (3, 10) into (3, 10+16 = 26) and so on, thereby reversing signs on *all 3* members (an *odd* number) of each triplet – suffices, in this case, to kill off all the ZD's we'd expect to see. Just as surprising, the triplet (1, 8, 9), "sterile" in the Sedenion context, provides the doublet from which the Assessors that triply repeat at B and E are fabricated.

A mystery, finally, is placed before us when the 7 (**s** > g^{-1}) tables, collectively dubbed "Sand Mandalas" after the Tibetan meditative drawings they resemble, are viewed in "flip-book" style: while the (*o*, 8, *o*+***s***) repeaters cluster inside the array, all other emanations are confined to the perimeter, with this square's edges pulling back in unison from the table's boundaries by one cell with each increment to ***s*** (while the "repeaters" reconfigure), until we're left in the last still-shot with a 2-cell-thick *cross* whose ends are joined up by the "repeaters" – which have rearranged themselves into diagonals.

It was this behavior (whose necessity was proven, after the fact of stumbling on it, explicitly via elaborate bit-twiddling in "BK2" on p. 16) which first suggested something like cellular automata at work. More peculiar still, if one *folds the table over* like a handkerchief, along both horizontal and vertical midlines, the L-terms in the "shrinking box" coalesce to form the O- and S- terms of the 6 Assessors in the *Sedenion* Box-Kites whose strut constants = the *differences* of the table's **s** and g^{-1} terms (i.e., 1 - 7); the "repeaters," meanwhile, coalesce to form the "forbidden doublet" of the Sedenion Box-Kite's **s** and *g*!

“Muster Master” and “Sand Mandala” Emanation Tables (*s* = 8 through 15)



The above graphics are reproduced from p. 15 of “BK2,” and should be read from left to right, top to bottom: the first “emanation table” is for **s** = 8; the last, for **s** = 15. Rows, columns, and cell contents show L-index numbers of the associated Assessors in hex (since values range from 1 to 15). Rows and columns are further arranged so that strut-opposites appear in “nested parentheses” format, in otherwise ascending order, as with the earlier Sedenion strut table. Since diagonals will always be empty of emanations, they are simply left blank; in all but the first table, the majority of non-diagonal cells are also unfilled, and these are left empty, but with light gray background. Cells which form into the 3 Box-Kites' (*o*, 8, 8+*o*) “repeaters” are daubed green; the others are painted red.

As one might expect, as the N in 2^N-ion increases, “carry-bit overflow” becomes even more pronounced, with patterns that look progressively more “fractal” than those shown: at high **s**, as cells become more numerous, boundaries become crenelated, and box-kites even break up into pieces. It is unclear what happens for very large N: does the density of ZD's → 0 as N → ∞? Would that imply (and quite counterintuitively) that high-**s** , high-N spaces become “tame,” perhaps with something like a Hilbert space as a limit; or, do they instead “merely” become unwieldy and chaotic? Abolutely nothing is know at present – including whether or not these are good questions!

The heading for the graphics indicates that the “Muster Master” is also on display. Clearly, it's the upper leftmost, **s** = 8, table: ironically, the same index plagued with ZD-sterility when it's the generator becomes the most “omnivoyant” of all configurations when it is “demoted” to subgenerator status in the next round of CDP expansionism. Precisely because it's *ex*cluded, all the Octonions are *in*cluded among the L-trips – leading to 7 Box-Kites each of whose Zigzag has an O-trip for lower indices . . . meaning “slippage” between U-trips and O-trips could be effected “ionically” or “covalently” between the Muster Master and *any* Sedenion Box-Kite, and in an“unobservable” manner, thanks to Zigzag involvement. But Sedenion S-trips can be Pathion L-trips, so “bonding” is subtle.

We could, in fact, describe the situation, using the argot of object-oriented programming, as perfect "design-hiding": for while the *o*'s in the ABC positions (and hence, the Zigzag) of each Sedenion Box-Kite exactly match the Zigzag L-trip of exactly one of the 7 component Box-Kites of the Muster Master, it is the *S*'s in the former's DEF positions which get promoted to L-trip status in the latter. Hence, Trefoil L-trips "resonate" with the U-trips of the lower-order Box-Kites – but only with those that share their singleton *o* with the Zigzag! The *Muster Master* Box-Kite which shares Zigzag L-trips with a given *Sedenion* Box-Kite cannot have its actions observed at all from the latter's purview.

This is not the whole story, though: in Sedenion Box-Kite IV, for instance, the S-trip (13, 11, 6) – which we'll write (A, D, e) to obey our casing convention – "resonates" with (f, d, b) of the Muster Master's Zigzag-match to Sedenion Box-Kite I, and *this* interaction *is* a Trip-Sync "observable." Below is a synthesis of data from the "Synchronization Table By Box-Kite and Sail" on the penultimate page of "BK3," and the component Box-Kites of the Muster Master, listed by their Sedenion Zigzag-matches' strut constants (shorthanded "ZM#" in the table). Note that vertex locations are rotated (®) in 4 of the 7 Muster Master components: this affects labeling (e.g., Trefoils might switch places and their cyclings shift in phase), but not triplet orientation, and so can be ignored.

Trip-Sync Observables in Muster Master / Sedenion Box-Kite Interactions

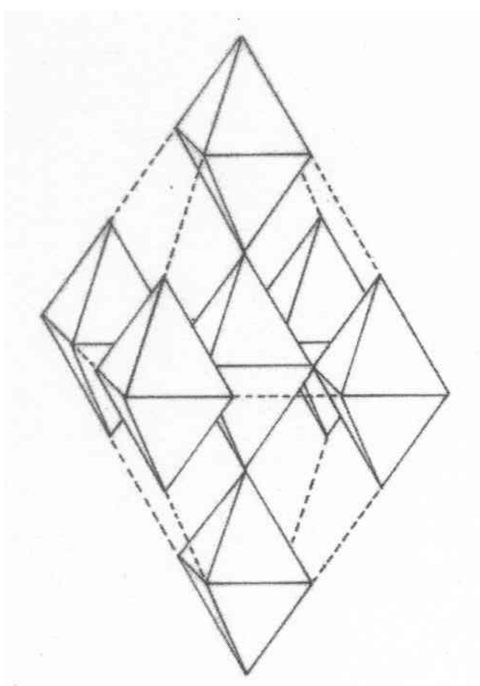
							MM L-trips Observable (or not) in BK#... As ...		
ZM#	**A**	**B**	**C**	**D**	**E**	**F**	**(ade)**	**(fce)**	**(fdb)**
I	3	6	5	13	14	11	*(3,13,14)*	*(11,5,14)*	*(11,13, 6)*
							II:fDB;**VII:**ADe	**II:**ADe; **IV:**fDB	**IV:**ADe;**VII:**fDB
							(IV:ABc)	**(VII:ABc)**	**(II:ABc)**
II ®	6	1	7	15	9	14	*(6, 15, 9)*	*(14, 7, 9)*	*(14,15, 1)*
							III:ADe; **IV:**FCe	**IV:**FdB;**V:**fDB	**III:**FDb; **V:**ADe
							(V:ABc)	**(III:ABc)**	**(IV:aBC)**
III ®	7	2	5	13	10	15	*(7,13,10)*	*(15,5,10)*	*(15,13, 2)*
							I:ADe; **IV:**AdE	**IV:**fCE; **VI:**fDB	**I:**fDB;**VI:**ADe
							(VI:ABc)	**(I:ABc)**	**(IV:AbC)**
IV	1	2	3	11	10	9	*(1,11,10)*	*(9, 3, 10)*	*(9, 11, 2)*
							V:FCe; **VI:**FdB	**V:**FdB;**VII:**FCe	**VI:**FCe; **VII:**FdB
							(VII:aBC)	**(VI:aBC)**	**(V:aBC)**
V	2	4	6	14	12	10	*(2,14,12)*	*(10,6,12)*	*(10,14, 4)*
							I:fCE; **VII:**AdE	**III:**FCe; **VII:**fCE	**I:**AdE; **III:**FdB
							(III:aBC)	**(I:AbC)**	**(VII:AbC)**
VI ®	4	7	3	11	15	12	*(4,11,15)*	*(12,3,15)*	*(12,11, 7)*
							I:FdB; **II:**AdE	**II:**fCE; **V:**AdE	**I:**FCe; **V:**fCE
							(V:AbC)	**(I:aBC)**	**(II:AbC)**
VII ®	5	1	4	12	9	13	*(5, 12, 9)*	*(13, 4, 9)*	*(13, 12, 1)*
							II:FCe; **VI:**fCE	**II:**FdB; **III:**AdE	**III:**fCE; **VI:**AdE
							(III:aBC)	**(VI:AbC)**	**(II:aBC)**

Before analyzing the results of the table above, some backgrounding concerning the Muster Master is in order, the better to make the results we'll be discussing seem all the sharper. For completeness' sake, here's a listing of the full set of the Muster Master's 14 Assessors, showing both L- and U- indices, with strut-pairs arrayed vertically:

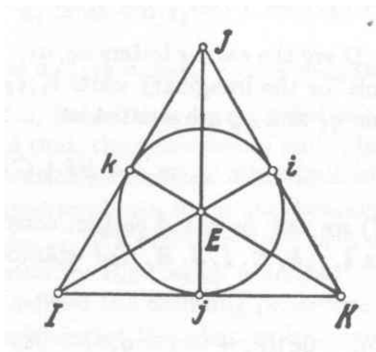
(1, 25)	(2, 26)	(3, 27)	(4, 28)	(5, 29)	(6, 30)	(7, 31)
(9, 17)	(10, 18)	(11, 19)	(12, 20)	(13, 21)	(14, 22)	(15, 23)

The utter simplicity of this deployment belies the complexities implied in the table. It would greatly aid intuition to have some way to picture the synergy between the 7 component Box-Kites of this highly symmetric "father" to the Pathion "Pléiades," whose 7 "nymph" Assessors with *o*'s for L-terms nicely echo the 7 "stars" with L-term *S*'s who form their strut-partners. The two descriptive pieces of this last sentence each will lead us to investigate a different graphical depiction. Let's begin, first of all, with a way to envision 14 Assessors, each in 3 of the 7 "cleanly partitioned" Box-Kites (by which I mean, no edges are shared between them), appearing thereby in twice 3 = 6 Sails. The following is lifted from a paper describing "octahedral fractal" models[12] – a topic which has no direct bearing here, although the graphic fits our themes perfectly.

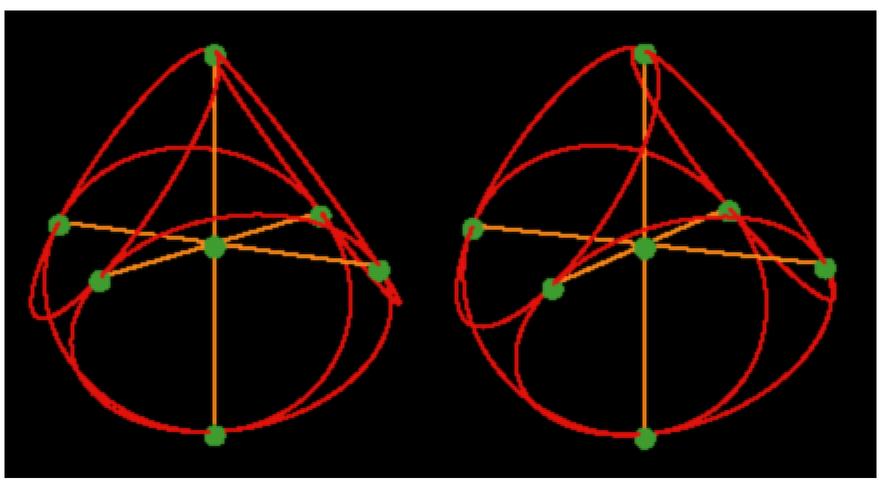
We can scan this as 7 stacks of 3 Box-Kites– 3 trios including the central one, and 4 more joined by the dashed lines. Assume stacking suggestive of a voltaic pile for the centrally coordinated sets: if the Assessor at the top of the vertical stack, for instance, is "A," it would be replaced in its position in the next-lower Box-Kite by its strut-opposite, "F," so that the sequence of descending labels for the trio would be this: A, F=F, A=A, F; similarly for the perpendicular lines of Box-Kites in the central horizontal plane. For the other 4 trios, assume vertices joined by dotted lines from the central squares are strut-opposites as well. There's a lot of rotating and translating you have to deal with to make a consistent labeling scheme; putting in indices, too, would induce a truly noxious degree of cluttering, so I won't even try. The idea, I hope, will suffice!



While this graphic, then, serves to present an image of the interconnectedness of Assessors in the Pléiades scheme, it gives no clear feeling for Assessor-level detail: a way to picture the indices without the distraction of 3-D would be a useful thing. Here, we can adapt the tactic, mentioned in passing at the end of the last section, of representing triplets as intersecting lines, via finite projective groups: a "stereo" version of of the PSL(2,7) layout of labeling schemes for the Octonions which was championed by the late great Canadian geometer Harry Coxeter. The interrelations among indices for Atlas and his Pléiades can be cleanly displayed in this manner, and "loop algebra" subtleties framed naturally, in their turn, by referencing these diagram-pairings.



PSL(2,7) expresses the symmetries of the simplest non-trivial finite projective plane, known by Fano's name. In projective geometry, lines and circles are equivalent, so this diagram should be interpreted as comprising 7 lines, including the apparent "circle" in the middle. The lines intersect in 7 points, the Octonion units: the cross-overs with no nodes or labels are illusions of the projection process. In this elegant rendering (from Ref. 1, p. 46), Quaternion units are in lower case (the standard "i,j,k") and make the central cycle. The (index-4) generator is at their circle's center, E. The triangle's vertices are just the products of their lower-case equivalents with E, so that lines through E are oriented from lower to upper case letters. Similarly, all three edges on the perimeter flow clockwise, while the central "i,j,k" flows likewise. The symmetry group of this 2-D configuration of 7 lines, each being intersected by 3 others (or equivalently, thanks to projective duality, 7 *points*, where each "cycle" is determined by 3 of them) is just 7 x 3 x 2^3 (the number of possible signing conventions per triplet), or 168. But this is precisely the total symmetry of "Box-Kite space" as well, since we have 7 of them, each with 4 3-cycles or "Sails"(which means they obey the 24-element rotational symmetry of their Octahedral container). If you have stereo glasses available, you can "see" this from Buckland Polster's Fano Plane graphic from <u>The Mathematical Intelligencer</u> **21** (1999) 38-43: here, the 7 "lines" are just the 3 Struts and 4 Sails, with the 7 "points" being the vertices plus the Zero (which we assume is "piped in" to the centers of each Assessor-plane "screen" by Strut-connected "cable TV"!)



What I have in mind here is another sort of "stereo," with the 14 Assessors in any given Pléiade being partitioned across two such layouts. Just as the PSL(2,7) triangle above can have its letters removed and relabeled by myriad different indexing schemes, with the arrows also having their orientations switched accordingly, so too with our "stereo Fano" manner of showing Pléiade symmetries. In the little table at the top of the next page, I'll just give the index-number equivalents for the letters, and assume you can figure out the O- trip and S-trip orientations without my help at this point.

“Stereo Fano” Synergies on Display (7 Pléiades + “Atlas”)

Pathion Strut Const.	“7 Nymphs” (3-cycles = mostly “Zigzags”) i	j	k	E	I	J	K	... turned into ...	“7 Stars” (Strut Opposites of Nymphs) i	j	k	E	I	J	K
I	8	14	6	13	5	3	11		9	15	7	12	4	2	10
II	8	15	7	14	6	1	9		10	13	5	12	4	3	11
III	8	13	5	15	7	2	10		11	14	6	12	4	1	9
IV	8	10	2	11	3	1	9		12	14	6	15	7	1	9
V	8	12	4	14	6	2	10		13	9	1	11	3	7	15
VI	8	12	4	15	7	3	11		14	10	2	9	1	5	13
VII	8	12	4	13	5	1	9		15	11	3	10	2	6	14
VIII	6	4	2	5	3	1	7		14	12	10	13	11	9	15

For all but “Atlas,” the 7 L-trips in each line of the lefthand listing split up so that 4 of the component Box-Kites are represented by their Zigzags, and 3 (all and only those containing an “8,” always situated at “D”) by their ADE Trefoils. This is the Pathions' equivalent of the “Strut Table” provided for Sedenions. The fact that a fully functional copy of PSL(2,7) can uniquely serve to integrate all a Pléiade's Box-Kites (with its “shadow” on the right providing “stereo vision” by housing, vertex by vertex, the lefthand terms' strut opposites) gives striking proof of the Pleiades' fundamental synergies as ensembles: they are by no means loose affilations of independent entities, “cat-herded” opportunistically. (Exercise: draw all 7 Kites for each line, properly orienting the edges!)

Now it is time to turn our attention, at last, to the “Trip-Sync Observables Table” on Page 14. The first thing to stress is that there is an exact, one-to-one correspondence between Sails on the Sedenion “receiving end” and the Pathion “sending end” of any transactions between them: tautologically, each ZM# indicates a Zigzag-to-Zigzag matchup of the “unobservable” kind; but there is also precisely one “unobservable” link-up, from each Muster Master component's Trefoil Sail L-trip, to some specific S-trip in some Sedenion Zigzag. And there are also precisely two “observable” connections from Trefoil-based Muster Master L-trips to specific Sedenion Trefoil S-trips. Such extreme orderliness and specialization of connection indicates we're dealing with a very stable, well-articulated platform for doing *some*thing.

But doing what, exactly? Here is where I see the next line of research, which will, I hope, be ripe for presenting at NKS 2005. Viewed in full generality, we have this for a setup: a latticework of connected Assessors, where Sedenion Box-Kites can “channel” a quartet of independent inputs from the Pathions' “higher plane”; these inputs are effectively binary, and there is one output, also binary. We have also seen, from many angles now, the critical role of 168-fold symmetry in all these workings. It is at this point that Rodrigo Obando's ongoing work[13] on applying Boolean function theory to NKS systems shouts for our attention: for there are 168 4-variable monotone functions, and the same number of complements, which he first indicated at NKS 2003 can provide the key to determining the classification of NKS behaviors displayed by arbitrary rules. His basic tactic (and its clear “resonance” with ZD setups) goes like this.

Split the bit representation of a rule into two "primitives." One string contains bits indicative of an initial input combination that has the central cell's value = 0. The other contains those telling what happens if said cell value = 1. For the (r=2) situation that clearly obtains with the Trip-Sync linkup of Pathions to Sedenions, the possible rules equal 2 raised to the (2^5 =) 32^{nd} power, which is more than 4 billion rules altogether – an awful lot to have to wade through if we've no initial notion of which ones are Class 1 (boringly homogenous outcomes), which are Class 2 (evolving into simply separated periodic structures), which yield up chaotic aperiodic patterns (Class 3), and which – the real "money shot" here – generate truly complex "Class 4" patterns of localized structures.

But Obando's work strongly suggests (with some specific examples to buttress his claims) that we can, indeed, predict what sort of behavior a rule will display, and even say which will be Class 4. First, partition a rule into "primitives," then see if their bit representations are Boolean monotones. Because the "primitives" have half the string each, this means we're dealing with the much more manageable count of 2^{16} logical expressions, or 65,536 possibilities, that each "half-rule" can display. If the primitives are properly chosen from the two "monotonous" sets of 168 – each created recursively, by bit-string *and*-ing and *or*-ing, out of Oneness and Nothing – we'll get Class 4 complexity.

But meanwhile, we have two sets of 168 ZD "primitives," on opposite sides of the Sedenion/Pathion "infobarrier": one, then, from each party to our "channeling." And each Box-Kite "edge current," if encoded as a string of 0's and 1's, can be uniquely specified by *precisely 16 bits.* Consider it this way (and there are other ways as well): any "successful" ZD product entails the involvement of 2 Assessors, which for the Sedenion "target" (and isomorphically for Pathion "broadcaster") means 4 bits for each *S* index, 3 for each *o*, and 2 extra bits "left over." Suppose we say these latter determine the diagonals – "\ or /?" – as specified at the two ends of the connecting edge. (N.B.: the edge sign's value itself is *not* an independent variable, so can't be counted!)

Now we need to tweak this (or is it "kludge things up"?) a bit. The problem is, stating just this much means we get *two* distinct strings for each "edge-current," depending upon which Assessor we write out first. But rather than get worked up about this double-count of strings, let's make use of this implicit positional information, in *this* way: if the first Assessor is "less than" the second (the *o* index is lower, say, or it has an earlier birthday: it doesn't matter *how*, just *that* it's "less than"), then assume they *both* employ the "/ " diagonal. If the first written down is the *greater*, however, assume " \ ". Next, use the first of our leftovers for a "switch bit" in this sense: if "0," do nothing; if "1," switch the diagonal used by the second in *sequence,* not size. Which leaves us with one last "extra" to interpret: I call this bit the "spigot," since I envision it as turning the "edge-current" on or off – something I have trouble imagining *not* playing a role!

And while the "168" in the ZD universe are most definitely *not* Boolean monotone functions by any stretch of the imagination – their fundamental workings, after all, are defined by XORing, which is *not* an allowable operation in monotone string-building – there most definitely is very good reason to expect a rich harvest of transformations between the two mathematical languages. Let me put this another way: even though I don't carry an AARP card yet, I still can remember when the idea of "Chaotic number theory" would have seemed too insane to bear contemplating; by the next NKS conference, I hope to make the case that the problem with that notion is, *it's just too tame.*

1 I. L. Kantor and A. S. Solodovnikov, Hypercomplex Numbers: An Elementary Introduction to Algebras (Springer-Verlag: New York, 1989) provides an elegant introduction (aimed at "smart high school students" as these were understood in Soviet-era Moscow) to the whole theme of hypercomplex numbers, all building up to the formulation, in Chapter 17, of Hurwitz's 1896 proof that "Every normed algebra with an identity is isomorphic to one of following four algebras: the real numbers, the complex numbers, the quaternions, and the Cayley numbers."
2 R. Guillermo Moreno, "The zero divisors of the Cayley-Dickson algebras over the real numbers," Bol. Soc. Mat. Mex. (3) **4:1** (1998), 13-28; http://arXiv.org/abs/q-alg/9710013. The key result was his finding that ZD's in the Sedenions obey a structure *homomorphic* to the Lie algebra indicated by the Dynkin Diagram "G_2"; but homo- is not iso- morphic, and only a single concrete instance of a pair of ZD's was given . . . which led to my own radically constructivist, almost "punk-rock"-like simplicity of 3 production rules and one octahedral-lattice-based "box-kite" graphic to summarize the workings of the entirety of 16-D ZD's. (Moreno notes, too, that "G_2" also implies a powerful scaling of results into higher 2^N-D number forms, since "G_2" – the automorphism group of the Octonions – is the "derivation algebra" of each 2^{N+1}-ion number system from the 2^N-ions which generate it by the "Cayley-Dickson process."
3 Robert P. C. de Marrais, "The 42 Assessors and the Box-Kites they fly: Diagonal Axis-Pair Systems of Zero-Divisors in the Sedenions' 16 Dimenions" (*"BK1")*, http://arXiv.org/abs/math.GM/0011260 ; "Flying Higher Than a Box-Kite: Kite-Chain Middens, Sand Mandalas, and Zero-Divisor Patterns in 2^N-ions Beyond the Sedenions" (*"BK2"*), http://arXiv.org/abs/math.RA/0207003; and, "Quizzical Quaternions, Mock Octonions, and Other Zero-Divisor-Suppressing 'Sleeper Cell' Structures in the Sedenions and 2^N-ions" *("BK3")*, http://arXiv.org/abs.math.RA/0403113.
4 R. E. Cawagas, "Loops Embedded in Generalized Cayley Algebras of Dimension 2^r, $r \geq 2$," Int. J. Math. Math. Sci. **28:3** (2001), 181-187. Prof. Cawagas' "Finitas" software, which he makes freely available, lets one calculate loops and other nonassociative structures rather easily in higher-dimensional contexts; he in fact found an equivalent manner of expressing my tabulations from BK1 in "loop language," which he presented in a paper of this year to the National Research Council of the Philippines, "On the Structure and Zero Divisors of the Sedenion Algebra"; the correspondence which ensued after he communicated this to me proved delightfully engaging, and was the direct trigger for "BK3."
5 Trickier "squares rules," though, abound, as do esoteric algebras concerned with them, such as "Pfister forms"; A. R. Rajwade, in fact, has devoted a whole book to them: Squares (Cambridge U. P.: Cambridge UK, 1993). Using his convenient shorthand, the standard rules have form (2^n, 2^n, 2^n) where $n < 4$; for 16-D, though, two 16-D numbers, comprising 16 squares each, generally have a product representable by 32 squares (16, 16, 32). But stranger results are also possible: (9, 16, 16) and (10, 10, 16) also work – which may "mean something very significant" as far as Sedenions are concerned (except, whatever it is, no one's discovered it is yet!).
6 Jaak Lõhmus, Eugene Paal, Leo Sorgsepp, Nonassociative Algebras in Physics (Hadronic Press: Palm Harbor FL, 1994), p. 242.
7 The clearest exposition I've found of this "unobservability principle" – which the authors actually relate to St. Augustine's views on the origins of Space and Time! – is contained in the string-theoretic (but not yet M-theory aware) text by Q. Ho-Kim, N. Kumar, and C. S. Lam, Invitation to Contemporary Physics (World Scientific: River Edge NJ, 1991), 442-446. (Their discussion also indicates why 1-D "strings" – but not yet why "p-branes," $p > 1$ – in the first place!)
8 A. D. Sakharov, "Vacuum quantum fluctuations in curved space and the theory of gravitation." Dokl. Akad. Nauk SSSR, **177:1** (1967), 70-71 (Russian); English translation in Sov. Phys. Dokl. **11:3** (1968), 381.
9 Agnes Arber, The Mind and the Eye (Cambridge U.P.: Cambridge UK, 1964), 41: "Every grade can be traced between remote analogies, and analogies which are so close that they pass into identities, and – paradoxically enough – it is often the remote analogies which have the greatest value, while it is the close analogies of which we have to beware. The misuse of an analogy by pressing it to the point at which it is confused with an identity, is one to which *biological* thought is peculiarly liable." Where Ms. Arber says "biological," though, we can just as well speak of any heavily qualitative and/or speculative area of science (viz., cosmology) where metrics are hard to come by.
10 Robert Ray Britt, "The New History of Black Holes: 'Co-Evolution' Dramatically Alters Dark Reputation," www.space.com/scienceastronomy/blackhole_history_030128_1.html (January 28, 2003)
11 For completeness, I list in the "Strut Table"'s second column a set of indices I call "Goto numbers" in BK1, 4 per Box-Kite (and 1 per Sail). These refer to the 7 "deformed" Octonion copies the Sedenions contain, which I variously called "Moreno counterfeits" and "automorphemes" (the latter being a name only someone overly fond of Lévi-Strauss's structuralism could love). It was these which, by other means, Cawagas showed corresponded to instances of a hitherto unrecognized variety of ZD-harboring "loop." Each of these 7 loops contains 4 "Co-Assessor trio" 6-cycles, each of which appears in exactly one Box-Kite as a "Sail." (Making a "remote analogy," think of a Box-Kite, then, as a tetrad of parental chromosomes, with the "Zigzag" as the father's "Y" and the "Trefoils" as the 3 "X's.")
12 E. Battamer, "The fractal octahedral network of the large scale structure," http://arxiv.org/abs/astro-ph/9801276, p. 2. The caption on the graphic reads: "7 small octahedra inside a large octahedron when size ratio =3."
13 Rodrigo A. Obando, "Partitioning of Cellular Automata Rule Space," Complex Systems **14** (2003), 1-14.