Placeholder Substructures I: The Road from NKS to Scale-Free Networks is Paved with Zero-Divisors

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Zero-divisors (ZDs) derived by the Cayley–Dickson process (CDP) from N-dimensional hypercomplex numbers (N a power of 2, and at least 4) can represent singularities and, as $N \to \infty$, fractals and thereby, scale-free networks. Any integer >8 and not a power of 2 generates a metafractal or *sky* when it is interpreted as the *strut constant* (S) of an ensemble of octahedral vertex figures called *box-kites* (the fundamental ZD building blocks). Remarkably simple bit-manipulation rules or *recipes* provide tools for transforming one fractal genus into others within the context of Wolfram's Class 4 complexity.

1. Introduction: A sky for box-kites

Within a generation of taming imaginary numbers, William Rowan Hamilton etched his famous $i^2 = j^2 = k^2 = ijk = -1$ equation on the bridge he was crossing, thereby giving us quaternions. Eventually, the tools of vector calculus were derived from their workings. Within months, Arthur Cayley and John Graves independently generalized his four-dimensional (noncommutative) complex quantities to the eight-dimensional (nonassociative) octonions. But before the century was out, attempts to extend hypercomplex number systems indefinitely, by the Cayley–Dickson process (CDP) dimension-doubling algorithm, met with an impediment. In 1896 Adolf Hurwitz proved that the next re-doubling (to the 16-dimensional sedenions) would continue the pattern of giving up something familiar as the price of admission; only this time, the very notion of a norm and hence, of division algebra itself, would be forfeit [1]. The necessity of zero-divisors (nonnull numbers with zero product) in all higher 2^N-ions, $N \geq 4$, almost instantly squelched the ambitions of number theorists with such thoroughness that the next numbers (32-dimensional) were never even given a name, much less investigated seriously. Yet these *pathions*, as we will term them (as in

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"pathological," which is what such entities were deemed) are precisely those which hold the key to underwriting and algebraically generalizing fractals with number theory.

There is much more than irony in the fact that fractals themselves were similarly dismissed as "monstrosities" only to be later tamed and rendered commonplace by Benoit Mandelbrot [2]. For, thanks to their attribute of being scale-free (as statistical distributions), fractals buttress the recent theory of complex networks [3], which can be given its own number-theoretic basis (making it henceforth less reliant on empirical methods) by dint of zero-divisor (ZD) ensembles. Such a network-enabling metafractal (each of whose infinite points is associated with one among an infinite number of orthogonal lines of ZDs) we call a *sky*, for it is where box-kites fly. *Box-kites* are ensembles of six orthogonal planes, each represented by a vertex on an octahedral frame, and are the basic molecules of ZD structures. Their workings (and manner of embedding in skies, which we will visualize *via* spreadsheet-like "emanation tables" or ETs) will be our first concern. Once we understand their interrelations, we will be able to generate the simple rules that use the bit-string of a box-kite ensemble's integer signature to fill or hide ET cells, in a manner implying approach to the "fractal limit."

Our argument, like Caesar's Gaul, naturally divides into three parts. In this Part I, our focus is on the octahedral box-kite representation, which can organize all the seminal concepts of ZD theory in the 16-dimensional context in which it first emerged. We will end with a first foray into 32 dimensions, fabricating the small set of fundamental tools needed to keep expanding our agenda recursively into 64, 128, 256, and then infinite dimensions. In Part II, we will introduce the spreadsheet-like means of representing ensembles of box-kites with ETs, which lead to graphics more suggestive of colored quilts than any regular figures Euclid might have known. Their resemblance to classical fractals will soon be evident, and culminate in our construction of the simplest ("Whorfian") sky, whose basic building blocks only first emerge in 32 dimensions. Finally, in Part III, we will construct the component parts of the algorithm which lets us build an infinite-dimensional metafractal sky that is unique to any integer >8 and not a power of 2 (which is how we can justify calling our study a branch of number theory—but one based on bit-string-suggestive placeholder structures, with powers of 2 playing a role roughly akin to that of prime numbers in the quantity-fixated contemplations of which the Greeks were so fond). Here, we conclude with what literally boils down to "cookbook instructions" for building arbitrary skies, allowing for NKS-style movements between them using the recipe theory.

2. Preliminaries: Cayley–Dickson process and what zero-division means

We use the most general CDP labeling scheme: units are subscripted from 0 (reals) to $2^N - 1$; the product (up to sign) of any two has a subscript equal to the XOR of the producing terms' subscripts. Call $\mathbf{G}$ the index of the CDP generator of 2^{N+1}-ions from 2^N-ions. Then $\mathbf{G} = 2^N$ (8 for sedenions), and we have Rule 1: For 2^N-ion index $L < \mathbf{G}$, $i_L \cdot i_G = +i_{(G+L)} \equiv +i_{(G\,\mathrm{XOR}\,L)}$. Once we have at least one associative triplet (trip) to work with, we can invoke Rule 2: Writing in cyclically positive order (CPO) so positive units multiplied left to right always result in another, creating a new trip by adding $\mathbf{G}$ to two terms of another reverses the order: $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \to (\mathbf{a}, \mathbf{c} + \mathbf{G}, \mathbf{b} + \mathbf{G})$. These two rules (plus a Rule 0 which says trips in 2^N-ions remain trips in all higher 2^{N+k}-ions, $k > 0$) completely describe trip-making, hence CDP.

Examples

Appending a unit i_2 to the standard complex plane, Rule 1 gives us $i_1 \cdot i_2 = +i_3$, that is, the usual i, j, k of quaternions (2^2-ions). Performing one further iteration with $\mathbf{G} = 4$, Rule 1 applied to the singleton Q-trip $(1,2,3)$ yields $\mathbf{G} - \mathbf{1} = 3$ O-trips $(1,4,5)$; $(2,4,6)$; $(3,4,7)$. By Rule 2, we also derive three more: $(1,\mathbf{2},\mathbf{3}) \to (1,\mathbf{7},\mathbf{6})$; $(2,\mathbf{3},\mathbf{1}) \to (2,\mathbf{5},\mathbf{7})$; and, $(3,\mathbf{1},\mathbf{2}) \to (3,\mathbf{6},\mathbf{5})$. Altogether, then, CDP says octonions contain $1 + 3 + 3 = 7$ O-trips, that is, the *trip count* for $N = 3$, or just Trip_3, a number derived independently by simple combinatorics. To form an associative triplet, pick one unit from those available, then pick a second from those remaining, and then divide by six to allow for all the permutations that could have led to the designated triplet being fixed by two selections. (Hence, the sedenions, with $N = 4$, have $(2^N - 1)(2^N - 2)/3! = 35$ S-trips; pathions, with $N = 5$, have 155 P-trips; $\mathrm{Trip}_6 = 651$, and so on.)

Remarks

The unique Q-trip can be rotated to bring its units' indices into ascending counting order (ACO) as well as CPO: $(1,2,3)$. Two of the seven O-trips, though—$(1,7,6)$ and $(3,6,5)$—are "bad trips." Inspection of our examples indicates that of the three new trips resulting from applying Rule 2 to the Q-trip, only one—O-trip $(2,5,7)$—will be "good" in turn. Corollarily, starting with the sedenions, Rule 2 will produce two good and one bad trip for each bad trip fed into it: for instance, $(1,7,6) \to (1,14,15); (7,9,14); (6,15,9)$. Simple algebra readily persuades us that $\mathrm{Trip}_{N+1} = 4 \cdot \mathrm{Trip}_N + 2^N - 1$; further, that bad trips among the 2^{N+1}-ions $= 2 \cdot \mathrm{Trip}_N$, and good, this number augmented by the Rule 1 contribution, or $2 \cdot \mathrm{Trip}_N + \mathbf{G} - 1$. Hence, as $N \to \infty$, good and bad trips approach parity.

3. Getting started: Box-kite exclusion rules

Treat the eight triangles of an octahedron as like a checkerboard, with those of the same color meeting only at vertices (Figure 1). But instead of colors, envisage four are *sails* (jibs X, Y, Z, W, made of mylar, say), while those on the faces opposite them are *vents* (through which the wind blows). The three pairs of opposite vertices are separated by *struts*, the wood or plastic doweling in real-world kites. Such *strut-opposites* are the only pairings of vertices which are not mutual "divisors making zero" (DMZs): traversing any of the 12 edges always "makes zero," as the ends of each host two (mutually exclusive) DMZ pairs.

For this to make sense, we must explain what we mean by "traverse" in this context. At each vertex are two imaginary units of different index (one, as we will show, always with index less than **G**, say A or B, while the other is always of a greater index H or J). The edge between any two box-kite vertices which are not strut opposites always connects two such dyads whose possible products include zeros. Their general product consists of four components, created by simple term-by-term multiplication. With proper sign choices (a crucial matter to which we will return shortly), the product of the units indexed B and H will exactly cancel that of the units indexed A and J; likewise, the product of those indexed A and B will exactly cancel that of units indexed H and J. (Note, in this last case, the high bits of H and J cancel by XOR, whereas A and B, being less than **G**, lack these high bits to begin with.) Also, we say two sets of DMZ pairs frame each edge, since each vertex represents one among six orthogonal planes, each spanned by a pair of hypercomplex axes, whose diagonals house all (and only) "primitive" ZDs.

For the case just given, we write $(i_A + i_H)$ and $(i_A - i_H)$, with one of these providing two of the four terms in the zero product made with a (B, J)-indexed diagonal. We will see in Theorem 3 that two of the four possible choices of diagonals to multiply together will always have zero product, and determining which pairings make zero is where signing comes in. Put another way, since CDP-generated imaginaries square to

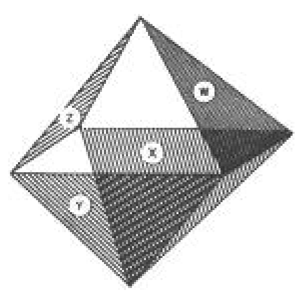


Figure 1. Box-kite, unadorned.

(-1) tautologically, a ZD is "primitive" only if comprised of the sum or difference of two such imaginaries. (DMZ also signifies "dyads making zero.") *Traversing* an edge, then, entails multiplying arbitrary points on suitably matched diagonals, one chosen from each terminating dyad, allowing some choice (since the order in which we multiply them has no effect on whether we get a zero or not) as to which end we think of as the starting point.

The reason for this image will become clear when we consider chaining such traversals: certain sequences of vertex-to-vertex multiplication effectively take us on closed tours of some subset of the box-kite's pathways, in such a way that we always make zero at each step in our circuiting. There are only a few such closed circuits, falling into a small number of types. These have surprisingly rich and distinctive properties as we shall see. Our focus here, though, will be primarily on the three-vertex, six-cyclic, kind, which we have already given a name: the sail. (The third vertex is "emanated" by the first two ZDs, since the zero the latter make is in fact the sum of two oppositely signed copies of this third dyad's terms. What makes the sail an algebraic whole is the fact, shown in Theorem 5, that any of its three edges, when traversed, emanates the third vertex in this sense.)

As the general reader is likely not a trained algebraist, some connection between such notions and those more familiar to computer-based thinking are appropriate. We note, then, that no notion is more fundamental than that of a solution, and the algebraic study of solution spaces can be seen as fundamentally devolving upon ZDs and imaginary quantities simultaneously. The very origins of the algebraic theory of equations demanded imaginaries as intermediary terms, as Cardano found when he discovered the general form of the cubic equation in the Renaissance. And as is well known, the intractability of quintic equations to extensions of such methods led to the classic group-theory-spawning studies of Abel and Galois. And here, we find the symmetry groups of equations devolving upon the structure of the zeros they contain.

Taking the step from calling the two terms $(x-b)(x-c)=0$ in a quadratic equation, say, mere factors, to legitimate ZDs in our sense here, is one in keeping with historically recent paradigm shifts in physics. The move from point particles to one-dimensional strings and multi-dimensional n-branes is roughly parallel to our own shift from ZD *points* to diagonal *lines* and exotica like the Seinfeld hyperplanes of our November, 2000, paper [5]. And just as one gets from points to lines by stringing an indefinite number of the former together in an ensemble, we will expand the singleton imaginary fundamental to Galois' thinking to the infinite array of such units afforded by CDP.

As CDP will be our starting point when our argument resumes, we allow ourselves one final comment before returning to it. Galoisian zeros offer a basis for modern representation theory, providing the starter kit (once embedded in Sophus Lie's continuous-groups treatment) for handling all linear forms (and even the "elementary" singularities of a nonlinear kind comprising René Thom's now-classical catastrophe theory). But the representation of the nonlinear, we claim, requires methods in the general case like those we adumbrate (and, to some small degree, actually implement) herein. And, knowing nothing more than the CDP rules, we can exclude some very general cases.

Theorem 1. Neither dyad can contain only indices $<\mathbf{G}$.

Proof. If both dyads contain only octonions, their product will contain only octonions, which have no ZDs. By induction, no ZDs new to the 2^N-ions generated by $\mathbf{G}$ can emerge from products of terms with indices $<\mathbf{G}$. If only one dyad has both terms' indices $<\mathbf{G}$, the term $\geq\mathbf{G}$ in the other will have unequal products with the first dyad's two terms. But, by XOR, these will also be the only terms with indices $\geq\mathbf{G}$, preventing cancelation of either. ■

Theorem 2. For any 2^N-ions, the unit indexed by $\mathbf{G}$ cannot belong to a ZD dyad.

Proof. Given trip (a, b, c) and binary variable $(\text{sg}) = \pm 1$, write $(i_a + i_G)$ below, and $(i_b + \text{sg} \cdot i_{G+c})$ above, and multiply term by term. The products by i_G are $-i_{G+b}$ and $\text{sg} \cdot +i_c$ (Rule 1). Products by i_a, however, are $+i_c$ (given) and $\text{sg} \cdot +i_{G+b}$ (Rule 2). Depending on sg, one or the other, but not both, pairs will cancel. ■

Remark. When N increases, former generators form ZDs. Starting with the pathions, though, a condition we term *carrybit overflow* can arise, wherein indices sufficiently large to play different roles for different N cause the breakdown of zero-division behavior in many cases where we would expect it, and it is precisely because of this that the metafractal skies will arise.

Theorem 3. Suppose two dyads each contain one unit whose index $>\mathbf{G}$, written in upper case, and another whose index $<\mathbf{G}$, written lower case. Then if $(a + A) \cdot (b + B) = 0$, $(a + A) \cdot (b - B)$ does not, and *vice-versa*.

Proof. Given the zero result, the *aB* and *Ab* terms, however signed, must cancel, as must *AB* and *ab*. But changing only one sign in one or the other dyad reverses sign on one unit only in each pair. ■

Remark. Clearly, multiplying either dyad by a real scalar k has no effect on ZD status, which is why these ZDs are best thought of as *diagonal line elements*. Represent ZD diagonals $k(x + X)$ and $k(x - X)$ in the obvious graphical manner as `(X,/)` and `(X,\)` respectively. It is easy to show these corollaries:

$$\texttt{(A,/)}\cdot\texttt{(B,/)} = 0 \Leftrightarrow \texttt{(A,\textbackslash)}\cdot\texttt{(B,\textbackslash)} = 0$$
$$\texttt{(A,/)}\cdot\texttt{(B,\textbackslash)} = 0 \Leftrightarrow \texttt{(A,\textbackslash)}\cdot\texttt{(B,/)} = 0$$

Theorem 4. Two ZD diagonals spanned by the same imaginary units cannot be DMZs.

Proof. $(a+A)(a-A) = +Aa+1$ on top, and $-1-aA = -1+Aa$ on the bottom. But $2Aa \neq 0$. ■

Remark. This is the exact opposite of the (nonCDP) ZDs found in quantum mechanics (QM), where the mutual annihilation of the idempotent projection operators $(1 \pm m)/2$, m a Pauli spin matrix (so that $m^2 = +1$, $m \neq \pm 1$), serves to define observables. There is a sense, however, in which the QM case is a degenerate form of the behaviors being examined here: this would take us off topic, but those interested in such questions should see [4].

Here is a final exclusion rule, alluded to above: ZDs living in planes associated with strut-opposite vertices can never be DMZs. This is a deeper result than those just presented, and is a side-effect of the general production rules, to which we turn next.

4. Completing the schematic: Box-kite production rules

In the sedenions, the most general production rule allows anything not excluded by the rules above: pick any octonion of index O (seven ways), then any pure sedenion with index $S > \mathbf{G}$ not the XOR of O with $\mathbf{G}$; the resulting dyad spans one of 42 planes, that is, the 42 "assessors" of [5], where the other production rules are also described for the first time. For higher 2^N-ions, restrictions due to carrybit overflow arise, but the approach just limned still generates all candidates for "primitive ZD" status.

From this rule to box-kites is but a short distance. Since each sedenion assessor will have an octonion in its dyad of units, clearly we can arrange their 42 planes into seven clusters of six, each of which excludes a different octonion from its set. Call the octonion index in each dyad (and, more generally, the 2^N-ion index $<\mathbf{G}$) the L-index, and its partner the U-index, and write their respective units i_L and i_U. Since each octonion is involved in precisely three O-trips, the six L-indices in each cluster can be uniquely fixed in two steps. First, assign each pair whose XOR is the excluded index to one of the three struts, which is necessary and sufficient to guarantee that their product does not appear in the set. (From what we have already seen, even without specifying what their U-indices are, this is tantamount to guaranteeing that no pairing of their assessors' diagonals can be DMZs.) Call the excluded index **S**, for "strut constant" and "signature." The last parenthetical remark tells us it somehow fixes the U-indices of the assessors in its cluster, we show how in section 5. For now, the more immediate problem of fixing L-indices leads us to perform our second step.

Orient the strut-opposite L-indices so that **S** is the central label in the PSL$(2,7)$ triangle which represents the smallest finite projective group, also called the Fano plane. Its three vertices, three sides' midpoints, and center each host an octonion index, in seven oriented lines of three points each, one per O-trip (in 480 possible configurations). Then, assign the L-index trip in the circle threading the sides' midpoints to vertices A, B, C on a box-kite, in CPO order. The strut-opposite L-indices will then appear as the vertices which terminate the lines extending from their L-index partners through **S**. We label them, in "nested parentheses" order, F, E, D respectively.

By removing the central label, four of the seven lines in PSL$(2,7)$ remain connected: that of the central circle just discussed, whose L-indices form the triplet (a,b,c); and, the three along the triangle's sides, which the diagram tells us have CPO labels (a,d,e), (f,d,b), and (f,c,e). We show in section 5 how these form the bases of the two types of sails: the singleton zigzag at vertices (A,B,C), and the three trefoils which each share a different vertex with the zigzag, and demonstrate their special properties which hold for all box-kites in all 2^N-ions. For now, we introduce the other production rules, the first of which motivates the very idea of a sail.

Theorem 5. If assessors (U,u) and (V,v) form DMZs, then a third assessor (W,w), forming DMZs with both, can be created by setting the index of W to $|U \text{ XOR } v| = |u \text{ XOR } V|$, and that of w to $|u \text{ XOR } v| = |U \text{ XOR } V|$.

Proof. Without loss of generality, assume L-index trip (u,v,w) is CPO, and the dyads of the given DMZ have the same internal signing, that is, $(u+U)\cdot(v+V) = 0$. We then have $u\cdot v = -U\cdot V$; but $(u,v,w) \to v\cdot w = +u$, so $w = -U\cdot V$ as well. Using sg as before, write $W = \text{sg}\cdot[+Uv] = \text{sg}\cdot[-uV]$. Consider $(v+V)\cdot(w+\text{sg}\cdot W)$, the former written under the latter. (The same basic argument applies if we choose $(u+U)$ instead and write it on top.) Multiplying left to right by v, we have $+u$ and $\text{sg}\cdot(v\cdot Uv) = \text{sg}\cdot(-v\cdot vU) = \text{sg}\cdot(+U)$. Next, $V\cdot w = V\cdot(-UV) = V\cdot(+VU) = -U$. And $V\cdot\text{sg}\cdot W = V\cdot\text{sg}\cdot(-uV) = \text{sg}\cdot(-u)$. Set $\text{sg} = (-1)$, and all terms cancel. ■

Remark. Two assessors in the same box-kite *emanate* a third whose L-index completes the L-trip implied by the other two. (The zero they make is also a sum of oppositely signed copies of the emanated assessor.) Such a triad defines a *sail*.

Caveat. The proof as given is complete within the sedenions, which display alternativity: $(aa)\cdot b = a\cdot(ab)$. Generalizing to higher 2^N-ions, however, requires a further result derived later, namely: all three index-triples involving one lowercase and two uppercase letters, each from a different assessor in a sail, are trips, so that even associativity among units combined as required in our proof is guaranteed.

Theorem 6. If $(U+u)$ and $(V+\mathrm{sg}\cdot v)$ are DMZs, then so are their *twist products* $(U+\mathrm{sg}\cdot v)$ and $(V-u)$.

Proof. By direct multiplication, the DMZ given requires $U\cdot v = -u\cdot V$ and $u\cdot v = -U\cdot V$. The proposed DMZ resulting from the twist would require $V\cdot v = u\cdot U$, and $u\cdot v = V\cdot U$. But the second requirement is identical to what we know is true in the given DMZ. As for the first requirement, multiply the terms of the other known relation on the right by $u\cdot v = w$ to obtain it. ■

Remark. A few months after [5], and quite independently of it, Raoul Cawagas obtained the same listing of primitive ZDs in the sedenions. He obtained it by programmatic exploration, using his Finitas software, of *loops* (which do for nonassociative algebras what groups do for their more orderly cousins) [6]. In our correspondence, it became clear that twist products in fact connect ZDs which, while necessarily in different box-kites, are always in the same loop. Each of a box-kite's four sails "twist" to different loops, while each Cawagas loop meanwhile partitions into four sets that correspond to sails in different box-kites, indicating an interesting duality between the two quite differently framed approaches and their objects. For the surprising symmetries inherent in twist product patterns, see the discussions surrounding the Royal Hunt and Twisted Sister diagrams in [7, pp. 14–16]. See [5, pp. 11–21] for full exposition of all the production rules being discussed here, and pp. 15–16 for the twist product rule in particular.

Coming Attractions. Once the workings of the strut constant are understood, the reader will find it trivial to prove that opposite sides of any of the three orthogonal squares comprising the box-kite's octahedral framework twist to the same box-kite, while adjacent sides twist to different ones. (The trick is to realize that, for all assessors in any given box-kite, the XOR of their dyads' indices is $\mathbf{G}+\mathbf{S}$.) This suggests that the three squares, which we will call *catamarans* (*tray-racks* in earlier papers), are as fundamental as the four sails, but for the different purpose of long-distance navigating; we conventionally name them for the tall masts planted orthogonally on the transverse frames joining their hulls: *AF* for catamaran *BCED*, for example. Catamarans also come in pairs: as will also be clear when edge-signs are described, tracing the perimeter of a catamaran only engages four of the eight ZD diagonals in the assessor planes being passed through, whereas tracing a sail twice will engage all six residing in its three assessors. Catamarans, then, with their square perimeters and coming in pairs, could easily be represented by traditional box-kites, as opposed to our surreal octahedral ones.

Lemma. As every L-index appears in two distinct L-trips in a given box-kite, likewise every assessor appears in two distinct sails.

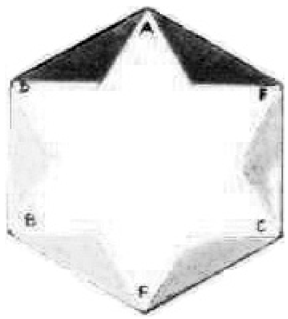
Remark. Full XOR-based calculation requires knowledge of the U-indices, which requires, in turn, a better understanding of the special role played by the strut constant, which is our focus in section 5.

5. Strut constants, sails, and "slipcover proofs"

The debt to the groundbreaking work of R. Guillermo Moreno is evident: his 1997 preprint (appearing in a journal the following year [8]) revivified the long-neglected study of ZDs. Moreno's now-classic paper culminated in a formal demonstration that the algebra of ZDs in the sedenions was homomorphic to the Lie algebra G_2, itself the automorphism group of E_8, in turn represented by the octonions, a result seemingly of interest only to physicists. Yet he also pointed out that this same G_2 is the derivation algebra of 2^{N+1}-ions from 2^N-ions, implying a deepset recursiveness in ZD structuring. We underscore that the Fano plane diagram for octonion labeling is likewise a potentially recursive tool: the 15 points and 35 lines, with seven lines through every point and three planes in each line, of the three-dimensional "projective tetrahedron" that schematizes sedenion labeling, is further cobbled together from 15 planes isomorphic to the PSL(2, 7) triangle. Higher labeling structures of $2^N - 1$ points, Trip_N lines, and so on, are readily imagined (albeit impossible to visualize). Our next results will be derived with virtually no tools beyond CDP and PSL(2, 7). We begin with a concretization of the discussion which introduced section 4.

Place the indices of the octonions so as to reflect their construction *via* CDP from quaternions: in the center of an equilateral triangle, place $\mathbf{G} = 4$; with apex extending above 12 o'clock, place $(1, 2, 3)$ at the midpoints of the left, right, and bottom sides, oriented clockwise, at 10, 2, and 6 o'clock in that order. Now its projective line, the only one drawn as a circle, represents the Rule 0 trip; lines oriented from the midpoints to and through the center are the Rule 1 trips; the sides, meanwhile, constitute the Rule 2 trips, and are all oriented in a clockwise manner, paralleling the flow along the Rule 0 circle.

Now replace $(1, 2, 3)$ with the letters (u, v, w), and replace the indices at the angles with the symbols $(u_{\text{opp}}, v_{\text{opp}}, w_{\text{opp}})$, as these are indeed u, v, and w's strut-opposites with respect to the central index, which we now replace with the symbol $\mathbf{S}$ for strut constant. The contents of the labels are indeterminate, but their choices must conform to the network of flows: clockwise along edges and central circle, and pointing



Strut Const	Assessors at Sedenion Box-Kite Vertices A	B	C	D	E	F
1	3,10	6,15	5,12	4,13	7,14	2,11
2	1,11	7,13	6,12	4,14	5,15	3,9
3	2,9	5,14	7,12	4,15	6,13	1,10
4	1,13	2,14	3,15	7,11	6,10	5,9
5	2,15	4,9	6,11	3,14	1,12	7,10
6	3,13	4,10	7,9	1,15	2,12	5,11
7	1,14	4,11	5,10	2,13	3,12	6,9

Figure 2. Hexagonal view of box-kite, with table of indices.

inward along the three lines bifurcating sides, then angles. In our initial setup, the octonion generator is indeed the **S** for the sedenion box-kite implicated. In a manner suggestive of how one pulls on and removes slipcovers from upholstery, we claim that this flow pattern is not only uniquely linked to representations with the central circle comprising the *L*-indices of the zigzag sail; but that one can, without inducing any tearing in the network of flows, "pull" any octonion index into the center, thereby representing a zigzag-centric depiction of the box-kite whose **S** is that index.

PSL$(2,7)$ has the (for our purposes) highly useful property that, in order to obtain a correct multiplication scheme for the seven units, it suffices to place (u,v,w) on any line, mark any of the remaining four points **S**, and then pass through it to designate the last trio of points as opposites of the first. Swapping in one point for another, then, is easy. The symmetries are so complete that we can take any point we please and make the case for all, so let us take seven in the $\mathbf{S} = 4$ diagram. Rotate the line containing 4 and 7 once in the downward direction to keep orientation: the vertical line, in top-down order, now reads $(3,7,4)$ instead of $(7,4,3)$. The central circle, if its clockwise orientation is to be kept, must be CPO starting with the left midpoint and ending in the 4 now on the bottom. There are just two possibilities: $(6,2,4)$ and $(5,1,4)$, but the former would reverse the flow on the right flank, reading $(1,2,3)$ from the lower right to the apex. As is readily checked, $(5,1,4)$ does exactly what we want and, as our table of sedenion assessors and strut constants makes clear (Figure 2), that is in fact the zigzag L-trip for the $\mathbf{S} = 7$ box-kite. The three-fold symmetry of the flows lets us choose any of the central circle's three nodes to house the 1 (which is the *L*-index of the *A* assessor in the box-kite in question) without affecting the pattern.

But suppose we wish to swap two lines instead of nodes: one of the trefoil sails on the sides for the central zigzag, say. This can be done, but at the cost of changing the flow structure. Keeping the convention of mapping (a,b,c) to (u,v,w), the trefoils (a,d,e), (f,d,b), and (f,c,e)

now get rewritten as $(u, w_{\text{opp}}, v_{\text{opp}})$, $(u_{\text{opp}}, w_{\text{opp}}, v)$, and $(u_{\text{opp}}, w, v_{\text{opp}})$. If we insist on maintaining the clockwise orientation of the central circle after replacing its labels with one of the three sets just written, the flows on the two sides not corresponding to the original (u, v, w) circle will be counterclockwise, and the two rays through the center starting from midpoints suffixed "opp" will now lead away from, not into, the angles. "Slipcover tugs" on the fabric of this network will send a trefoil linked to one **S** value into one linked to a different **S**, but the lack of the zigzag L-trip's threefold symmetry means it typically will be a different trefoil: if one tugs 2, say, into the center, the (a, d, e) trefoil of the $\mathbf{S} = 4$ box-kite is transformed into an (f, d, b). The side-effect of trefoils' broken symmetry explains why, in the table provided in Figure 2, zigzags are so regular (each O-trip appears as one exactly once), while trefoils behave so erratically (each O-trip appears thrice as a trefoil, but never in an evenly distributed manner; in fact, $(3, 6, 5)$ makes all its trefoil appearances as (f, c, e), while $(1, 2, 3)$ *qua* trefoil shows up only as (a, d, e)).

We mentioned earlier that PSL(2, 7) diagrams can be used to map not merely the interconnections of octonion labelings; but, given the recursive modularity of all higher 2^N-ions, heptads among the sedenions and beyond. We have used, for instance, "stereo Fano" diagramming to navigate and notate the structures found in the 32-dimensional pathions [9]. For the present, though, we need nothing so exotic: we just want to drop **G** into the center of diagrams already seen, not to replace **S**, however, but to supplement it. The reason for this is implicit in this critical theorem.

Theorem 7. The U-index of any assessor is the XOR of its L-index with $\mathbf{G} + \mathbf{S}$. For zigzag sails, all three edges connecting assessors conform, in the notation of the Remark to Theorem 3, to this pattern:

$$(\mathtt{M},/) \cdot (\mathtt{N},\backslash) = 0 \Leftrightarrow (\mathtt{M},\backslash) \cdot (\mathtt{N},/) = 0$$

For trefoil sails, the two edges including the assessor shared with the zigzag are governed, instead, by the other pattern from the same Remark, namely:

$$(\mathtt{M},/) \cdot (\mathtt{N},/) = 0 \Leftrightarrow (\mathtt{M},\backslash) \cdot (\mathtt{N},\backslash) = 0$$

Proof. Place (u, v, w) on the central circle of a PSL(2, 7) diagram, with all seven lines in zigzag flow orientation. Place $\mathbf{G} + \mathbf{S}$ in the diagram's central node, calculate the angle nodes, and readjust flows by Rule 2. The trips on the lines through the center are now oriented to point from angles to midpoints, and are written thus: $(\mathbf{G} + \mathbf{u}_{\text{opp}}, \mathbf{G} + \mathbf{S}, \mathbf{u})$; $(\mathbf{G} + \mathbf{v}_{\text{opp}}, \mathbf{G} + \mathbf{S}, \mathbf{v})$; $(\mathbf{G} + \mathbf{w}_{\text{opp}}, \mathbf{G} + \mathbf{S}, \mathbf{w})$. The side trips now flow counterclockwise. In CPO, from the top, we have these: $(\mathbf{G} + \mathbf{w}_{\text{opp}}, \mathbf{u}, \mathbf{G} + \mathbf{v}_{\text{opp}})$; $(\mathbf{G} + \mathbf{v}_{\text{opp}}, \mathbf{w}, \mathbf{G} + \mathbf{u}_{\text{opp}})$; $(\mathbf{G} + \mathbf{u}_{\text{opp}}, \mathbf{v}, \mathbf{G} + \mathbf{w}_{\text{opp}})$. We assert $U = (\mathbf{G} + \mathbf{u}_{\text{opp}})$ and $V = (\mathbf{G} + \mathbf{v}_{\text{opp}})$.

Multiply $(u+U)\cdot(v-V)$, the indices assumed to be subscripts of implied imaginary units:

$$\begin{array}{c} +v-(G+v_{\mathrm{opp}}) \\ \underline{+u+(G+u_{\mathrm{opp}})} \\ +(G+w_{\mathrm{opp}})-w \\ \underline{+w-(G+w_{\mathrm{opp}})} \\ 0 \end{array}$$

Now place $(u, w_{\mathrm{opp}}, v_{\mathrm{opp}})$ on the central circle of a trefoil-flow-oriented PSL(2,7); plunk $(\mathbf{G}+\mathbf{S})$ down in the center of it; multiply dyads as above, but with the same inner sign for the two products including assessor U, to complete the proof. ■

Corollary. As a side-effect of Theorem 7, we see that indeed each sail is comprised of four trips, thereby addressing the Caveat to Theorem 5. The zigzag, for instance, has one L-trip, (a,b,c), and three U-trips where high bits are lost by XOR, written (a,B,C); (A,b,C); (A,B,c). Each, supplemented by the reals, makes a legitimate copy of the quaternions. Moreover, any box-kite in any 2^N-ion domain will have sails with this property, it being a feature of our Rule 2 and the recursivity of PSL(2,7), rather than anything specific to (hence subject to the limits of) sedenion space. We merely use whatever **G** is appropriate in our "slipcover proofs" and choose a Rule 0 triplet to place in the central circle; or, equivalently, pick whatever trio of strut-opposites you care to use which satisfy the given **S** value. (There will be many to choose from as N grows large, so that we speak more of box-kite "ensembles" than box-kites as individuals.)

6. Intermezzo: Lanyards, semiotic squares, emanation tables, sand mandalas . . .

Moreno's $\mathbf{G_2}$, as a root system, has order 12 (14, if we are physicists and add in the two-dimensional Cartan subalgebra). The sedenions' 7 box-kites × 12 edges × 2 oriented flows along each gives 168 as the order of the second simple group, and is 12×14. The integer 168 is also a "magic number" in boolean function theory, where it arises in the context of the Dedekind problem as the number of four-variable monotone functions, with the same number of complements. The Dedekind problem concerns determining this rapidly growing number for arbitrarily large counts of variables. The relevance of such abstruse topics to rule-based approaches like cellular automata theory and Stephen Wolfram's *New Kind of Science* has been the focus of ongoing work by Rodrigo Obando [9, 10], who has employed their toolkit to create a kind of periodic chart of such rule-driven systems, highlighting which are not just Class 1 (boringly homogenous outcomes), Class 2 (evolving into simple periodic structures), or Class 3 (chaotic, aperiodic), but truly complex Class 4 patterns, like Wolfram's famous Rule 30. All of which has provided motivation to search ZD space for connections to similar modes of complexity, driven by similarly simple rules. We will conclude, as our abstract suggests, with just such a "recipe theory" with which to study infinite-dimensional metafractals. We will prepare for that journey by taking leave of simple box-kites in the sedenions, and go so far in this section as to find the first truly anomalous ensembles, with the "wrong number" of box-kites, in the 32-dimensional pathions.

We first want to indicate that even simple box-kites are much richer in structure than we have seen thus far. The two kinds of sails are instantiations of abstract objects we call *lanyards*, which are cords worn around the neck to hold knives or whistles, or strands of leather or plastic used to string beads and other gewgaws into charm bracelets. By chaining together DMZs, a lanyard threads the ZD diagonals of multiple assessors, attaining closure by returning to its starting point having touched all other ZDs in its ambit but once. They can be given signatures even more compressed than the notation developed in the Remark to Theorem 3: the zigzag is a six-cycle lanyard, which makes zero with all its sail's diagonals, in a sequence cyclically equivalent to the glyph inspiring its name: / \ / \ / \ . The trefoil has its own six-cycle signature: / / / \ \ \ . (The catamaran, meanwhile, has a four-cycle signature / / \ \ , which should clarify our earlier cryptic comments concerning it.)

Joins between diagonals with opposite inner sign are designated either with a [-] *edge symbol*, or if using color graphics, drawn in red. Exactly half a box-kite's 12 edges are red, consisting of the trio defining the zigzag, and those bordering the vent on the face opposite it (see Figure 2). Those edges linking similarly inner-signed diagonals are marked [+] or painted blue. The "blues," in fact, are two other six-cycle lanyards which contain all and only the blue edges, corresponding to the perimeter of the hexagon in the prior image. There are two because one connects only \ diagonals, the other only /'s. What might a simulation that switched edge-signs (converting the blues into zigzags, perhaps) correspond to in real-world processes?

Another type of six-cycle lanyard, the two-sail, five-assessor "bow-tie" whose doubly passed-through "knot" must be in the zigzag, has deep symmetry-breaking properties that may well play a fundamental role in string theory [4]. In general, each distinct lanyard (and there are others which the reader is invited to explore involving 4, 5, and all 6 assessors: see section III of [5] for more on some of these) creates a distinctive context of dynamic possibilities, rather than the semblance of an answer (as groups, too often, are assumed to offer). And, unlike groups, and as one might expect in 0- rather than 1-based algebraics, an identity element is nowhere in sight, with strut opposites the closest thing to inverses. Rather, the zero resultant defining the transit of each link in a lanyard's chain guarantees a certain design-hiding, suggestive of the unfolding of the multitude of zeros packed in a single seed-form or morphogenetic "vanishing point." Put another way, a theory of lanyards can be expected to provide some crucial ideas in an area currently lacking many: the study of nonlinear representations.

One area where a theory of nonlinear representations exists is in semiotics, where the French school of Jean Petitot, dually inspired by René Thom's catastrophe theory (CT) mathematics in the 1970s, and the structural rapprochement of semantics and syntax essayed by Algirdas Greimas, led to CT modeling of "archetypal nouns and verbs" on the one hand, and a delving beneath Chomskian notions of semantically impoverished recursiveness on the other. Using the CT toolkit, Petitot has built models of Greimas' own famous (among literary theorists, at least) semiotic square [11], as well as the equally celebrated (among, at least, the structuralism-savvy with anthropological leanings) canonical law of myths of Claude Lévi-Strauss [12]. But both these models can be embedded, in their turn, in ZD-based networks, which put the recursiveness back where it belongs: not at the level of meaning as such, but rather in how meaning-rich nodes can be constellated in an open-ended scale-free dynamic.

The key insights are two: first, to see the entire program of structural linguistics, whose origins were contemporary with the algebraic death-knell sounded by Hurwitz's proof, as a kind of CDP *manqué*, thanks to Hjelmslev's indefinitely extensible notion (seminal to Greimas' thinking) of form/content "double articulation" [13]. The second insight is to find semiotic substance in what would seem a lack of substance from our purview: by analogy with Shannon–Weaver self-correcting error-code theory, where most of the bits are expended in the error-trapping as opposed to the content being transmitted, we assume the area of interest is precisely the ZD-challenged parts of the ZD ensembles that convey the significances. Specifically, we find our metamodel of Petitot's CT reading of the semiotic square in the four units whose invariant workings govern the ZD-free strut-opposites: the real unit (which never appears in primitive ZDs), **G** and **S**, and their XOR (which, given that **G** always has only one bit, always to the left of all of the **S** bits, is also their simple sum, henceforth to be called **X**).

The index set $\{\mathbf{0}, \mathbf{G}, \mathbf{S}, \mathbf{X}\}$ defines the units of a quaternion copy, part of what Moreno terms the four-dimensional, ZD-free boundary of sedenion space, in which all box-kites, each with their own **S** and **X**, participate. We argue at length in [7] that the square with drawn diagonals ⊠ which Greimas made the centerpiece of his thinking, can be modeled by placing **G** on the diagonals, **X** on the horizontals, and **S** on the verticals. From the ZD perspective, the workings of the three modes of binary relations (contradiction between a seme and its absence, an on/off bit, along diagonals; contrariety of reciprocal presupposition along the horizontals; complementarity of implication along the verticals) can be formally encapsulated in the following relations. Pick two strut-opposite assessors, dubbing the zigzag dyad picked as (Z, z), and the vent dyad linked to it, (V, v). Then

$$v \cdot z = V \cdot Z = \mathbf{S}$$
$$Z \cdot v = V \cdot z = \mathbf{G}$$
$$V \cdot v = z \cdot Z = (\mathbf{G} + \mathbf{S}) \equiv \mathbf{X}.$$

These three relations were implicitly derived in passing while proving Theorem 7. Their importance (and not just for semiotic applications) suggests we give them a special name: the *three viziers*, to be short-handed henceforth (and rewritten in triplet form) as:

$$\mathbf{VZ1} : (v, z, S); (V, Z, S)$$
$$\mathbf{VZ2} : (V, z, G); (Z, v, G)$$
$$\mathbf{VZ3} : (V, v, X); (z, Z, X).$$

Getting from the atomic level of Greimas' square, to the small-worlds networking of literally thousands of mythic fragments, all interconnected and mutually unfolded in the four fat volumes of Lévi-Strauss's magnum opus, would require some surgery on the square, to allow for a competition between contraries, say, to be transmuted into an agreed-upon implication. But the simplest such transformative structures already require the pathions, and hence metafractal skies to fly in. Those are the vistas in front of ZD theory, only some of which we will get to see herein. One such surgery, corresponding to the type just suggested as an example, will in fact provide the first recursive iteration leading to the fractal limit. And, as we are about to see, it arises naturally, virtually as a built-in feature, of the ZD framework.

As anyone seeing a ZD diagonal on an oscilloscope screen would think, assessors are ultimately patterns of synchrony and antisynchrony between their dyadic terms. Assume assessor-unit synchrony breaks down, with dyads recast as strut-opposites. Each finds higher-index partners; the former **G** and **S** are transformed into pathion L-indices at **B** and **E**, the former **X** moves to center stage as the new **S**. 14 assessors ($\mathbf{G} - 1$, minus 1 more to exclude the strut constant) form into three box-kites sharing the **BE** strut. It is quite possible to draw this three-headed beast, but the many-headed hydras in higher 2^N-ions will not prove so amenable. It is time to explore a different mode of representing things, which means emanation tables (ETs). And it is time to deal with a new phenomenon whose crucial nature we have indicated more than once: for values of $\mathbf{S} \leq 8$, the ensembles associated with each such integer, contain seven, not three, box-kites, and are dubbed *Pléiades* for this reason. But the surprising graphical sequence revealed as one thumbs through a flip-book, whose successive pages display ETs linked to progressively incremented **S** values from 9 through 15, inspired a different name for these entities:

> Viewed in sequence, these tables suggest the patterns made by cellular automata; seen individually, they suggest nothing so much as the short-shelf-life "sand mandalas" of Tibetan Buddhist ritual, made by monks on large flat surfaces with colored sands or powdered flowers, minerals, or even gemstones. [14, p. 15]

The second sweep of our argument [15] will begin with these two themes as its ultimate agenda, pursuing them in accordance with our originally advertised intentions. Once we have classified types and characteristics of ETs through the 64-dimensional chingons, we will have all the basic patterns needed for crafting (and proving) the algorithmics of "Recipe Theory" in Part III [16].

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