Franklin Squares - a Chapter in the Scientific Studies of Magical Squares

Peter Loly
Department of Physics, The University of Manitoba,
Winnipeg, Manitoba Canada R3T 2N2
(loly@cc.umanitoba.ca)

July 6, 2006

Abstract

Several aspects of magic(al) square studies fall within the computational universe. Experimental computation has revealed patterns, some of which have lead to analytic insights, theorems or combinatorial results. Other numerical experiments have provided statistical results for some very difficult problems.

Schindel, Rempel and Loly have recently enumerated the 8th order Franklin squares. While classical *nth* order magic squares with the entries $1..n^2$ must have the magic sum for each row, column and the main diagonals, there are some interesting relatives for which these restrictions are increased or relaxed. These include serial squares of all orders with sequential filling of rows which are always pandiagonal (having all parallel diagonals to the main ones on tiling with the same magic sum, also called broken diagonals), pandiagonal logic squares of orders 2^n derived from Karnaugh maps, with an application to Chinese patterns, and Franklin squares of orders $8n$ which are not required to have any diagonal properties, but have equal half row and column sums and 2-by-2 quartets, as well as stes of parallel magical bent diagonals.

We modified Walter Trump's backtracking strategy for other magic square enumerations from GB32 to C++ to perform the Franklin count [a datafile of the $1,105,920$ distinct squares is available], and also have a simplified demonstration of counting the 880 fourth order magic squares using Mathematica [a draft Notebook]. Our early explorations of magic squares considered as square matrices used Mathematica to study their eigenproperties. We have also studied the moment of inertia and multipole moments of magic squares and cubes (treating the numerical entries as masses or charges), finding some elegant theorems, and have shown how to easily compound smaller squares into very high order ones, e.g. order $12,544\left(= 2^8 \times 7^2\right)$. At least two groups have patents on using relatives of Franklin squares for cryptography, while a group at Siemens in Munich using pandiagonal logic squares has another pending. Other possible applications include dither matrices for image processing and providing tests

for developing CSP (constraint satisfaction problem) solvers for difficult problems.

1 Introduction

This article is based on a poster and allied commentary given at NKS2006. The present contribution comes on the heels of the landmark count of Franklin squares on a chessboard by Schindel, Rempel and Loly [1]. During the conference I received a message from Miguel Amela [2] enlarging on this work. It also incorporates a response to some of the questions posed by visitors to the poster. Pasles [3] has provided a beautiful historical context for the resurgence of interest in Franklin squares, and Maya Ahmed [4][5] posed the question which lead to our effort to count the number of Franklin squares which used the full set of elements 1..64. This work has already been reviewed by Ivars Peterson [6].

Small magic squares are often encountered in early grades as an arithmetic game on square arrays (patterns or motifs). Classical magic squares of the whole numbers, $1...n^2$, have the same line sum (magic constant) for each row, column, and main diagonals:

$$C_n = n(n^2+1)/2 \tag{1}$$

This line sum invariance depends only on the order, n, of the magic square. The ancient 3×3 Chinese Lo-shu square of the first nine consecutive integers is the smallest magic square and apart from rotations and reflections there are no others this size or smaller:

4	9	2
3	5	7
8	1	6

(2)

where the magic sum is 15. For $n = 3$, $C_3 = 15$,as expected. The best statement that can be made about its age is 2500 ± 1500 years! While the middle figure may be most appropriate, the hardest evidence gives just the lesser [7], while legends [8] claim the older.

We briefly review the recreational aspect of magic squares before drawing attention to the simpler semi-magic squares, and to pandiagonal non-magic squares. These we lump together under the rubric of magical squares. Then we examine the scientific aspects of all these squares through applications in classical physics and matrix analysis. In fact it is partly through the coupled oscillator problem that the mathematics of matrices was developed. Through an elementary example in matrix-vector multiplication, which can be done at the high school level, we demonstrate a simple eigenvalue-eigenvector problem. Magic squares can then play a valuable role in modern courses in linear algebra.

2 Recreational Mathematics

At the recreational level magic squares are fun for all ages, as I found when introducing them to visitors during the summer of 2000 whilst volunteering at the "Arithmetricks" travelling exhibit at the Museum of Man and Nature in Winnipeg, Manitoba. Various types of magic squares have become a recreational pastime of amateurs, often very gifted individuals e.g. Albrecht Dürer, and Ben Franklin. In 2006 I gave talks at various grade levels on aspects of magical squares.

While there are several journals which publish results in this area of recreational mathematics, the rise of the World Wide Web now affords many of the actors a place to publish extensive work without recourse to the oft tedious rigours of peer review, the selectivity of editors, and a considerable time delay. Fortunately there are some real gems from these efforts, but there are obvious drawbacks for the longer term flowing from absence of refereeing and lack of permanent archiving.

However magic squares present difficult challenges for mathematicians and over the past few hundred years many famous mathematicians have contributed to our knowledge of them, including Euler.

Consider the number of possible arrangements of 1..9 in a 3x3 square after removing an 8-fold redundancy factor from rotations and reflections, $9!/8 = 45,360$, so that the current fad of Sudoku has plenty of scope. Constraints on row, column, or diagonals line sums sharply reduce the number of squares, and there is plenty of room for other types of alternative constraints to produce interesting squares.

3 Art and Design: Line paths in Magic Squares

Stephen Wolfram posed several questions at my poster, one concerning the line path which can be drawn through successive numbers of a Franklin square. This probably goes back at least a century to a time when their were many fewer magic squares than are available today. Some of these are shown in Clifford Pickover's recent book [9], and he includes one from Ben Franklin. If one considers the 880 fourth order magic squares enumerated in 1693 by Frenicle de Bessy [10][11], which have been classified by Dudeney into 13 types, then the symmetry of the line path may distinguish squares with more than the minimal constraints. Stephen had in mind an underlying group theory. With the advent of larger complete sets of magic squares more study in this direction would be timely.

Early in the 20th century the famed architect Claude Fayette Bragdon [12] used line paths from some order $3, 4$ and 5 magic squares as the basis of ornament for the interior and exterior of buildings, especially in Rochester, New York.

4 Where is the Science?

Why did a theoretical physicist get involved with magic squares? This was not a linear process. Having been blissfully unaware of them for my first five decades, an encounter with the Myers-Briggs Type Indicator© [13] scheme of personalities resonated with my background in mathematical structures, partly from research in solid state physics. Coordinate rotation matrices in classical and relativistic mechanics, combined with periodic boundary conditions for finite crystals soon lead to links with magic squares, while the psychological nature of the MBTI eventually made connections with early work of Carl Jung, which finally connected with Jung's [14] interest in Chinese patterns, especially the dichotomous yin-yang schemes of Feng Shui (a.k.a. *The Golden Flower*) and the I Ching.

The science begins whenever we go beyond a recipe approach for constructing a single square, e.g. counting or estimating the populations of various classes of squares, proving that none are possible in a given case, interpreting them as arrays of point masses or electric charges. There are more examples, but another early aspect of our own work focusses on the remarkable results associated with treating magic squares as matrices, in the context of linear algebra, i.e. solving sets of simultaneous equations, as well as some topics in classical physics.

5 Some Special Varieties of Squares

For a number of purposes it is important to recognize that there are several specially important variations on the theme of magic squares.

Firstly, **semi-magic squares**, which do not necessarily have the diagonals summing to the row-column line sum, some of which may be obtained simply by moving an edge row and/or column to the opposite side, e.g.

9	2	4
5	7	3
1	6	8

(3)

The removal of the diagonal constraints means more squares due to the smaller number of constraints, in this case there are eight more. Secondly, pandiagonal non-magic squares, which have the same magic line sum for all the split lines parallel to the main diagonals. We can illustrate pandiagonals by taking a non-magic serial square (having the consecutive integers fill row-by-row) and tiling a copy to its right (or left, or top or bottom):

1	2	3	1	2	3
4	5	6	4	5	6
7	8	9	7	8	9

(4)

The pandiagonals are $(1, 5, 9)$, $(2, 6, 7)$, $(3, 4, 8)$, $(3, 5, 7)$, $(1, 6, 8)$, $(2, 4, 9)$, together with the main diagonals. Observe that for a given order the number of row and column constaints is the same as the number of pandiagonal constraints. Serial squares exist for all orders, unlike magic squares which do not exist for $n = 2$.

A pandiagonal magic square is the combination of this pandiagonal property with the requirements of a magic square. These first occur in order 4, and of these 48 are found amongst the 4th order squares, with none existing (possible) for singly even orders $(6, 10, ...)$:

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

(5)

A third interesting type, namely associative (or regular) squares have the antipodal property:

$$a_{ij} + a_{n-i+1,n-j+1} = n^2 + 1; i, j = 1..n \tag{6}$$

Finally, our recent Franklin result [1], and the most-perfect pandiagonal magic squares of McClintock [15], for which Ollerenshaw and Brée's combinatorial count ranks as a major achievement, draw attention to a fourth type which have the same sum for all 2×2 subsquares (or quartets). Franklin Squares also have constant half row and column sums, and constant parallel bent diagonals [1].

Ollerenshaw and Brée [16] have a patent for using most-perfect magic squares for cryptography, and Besslich [17][18] has proposed using pandiagonal magic squares as dither matrices for image processing.

6 Counting Magic Squares - the role of Backtracking Computations

There are 880 distinct 4×4 magic squares of the first 16 integers, and the $275, 305, 224$ distinct 5×5 of the first 25 integers, the latter were first counted by computer in 1973 [19]. Already by order six they have become uncountable, and as a result only statistical estimates are then possible. Pinn and Wieczerkowski [20] performed a Monte Carlo simulated annealing computation for an estimate of $(0.17745 \pm 0.00016) \times 10^{20}$ for the 6×6, and a less accurate estimate of the 7×7.

Walter Trump [21] developed a more efficient hybrid backtracking Monte Carlo method and improved the accuracy of these estimates. He has also made good estimates of the number of various types of magic squares up to 10×10, a remarkable achievement. During summer 2003 some of my undergraduate students at Manitoba took Trump's 6×6 GB32 code, which uses 13 random cells, and converted it to C++. Matt Rempel finds that the C++ code runs about 10% faster than GB32 on the same PC in producing a sample of some $700, 000$ squares, and has begun to remove random cells, finding longer run times, but more accurate results, and a larger sample of magic squares. Dan Schindel has taken the ideas in the 6×6 code and constructed pure backtracking codes without random cells to count the known numbers of magic squares for the 4×4 and 5×5 cases. While we have previously been able to analyse the complete set of 4×4's, we can now begin to analyse the 5×5's. This gives us the ability to study a variety of interesting questions, e.g. their eigenproperties. Dan Schindel has also amended the 4×4 code to count the number of pandiagonal non-magic 4×4 squares, finding some three million.

A number of others have polished backtracking codes which can be run from their web pages. Meyer [22] has one of the best and it will find a stream of different squares for $n = 4, 5$ and 6.

A simplified demonstration of counting the 880 fourth order magic squares using Mathematica [a draft Notebook][23]. Our Mathematica code can be refined and we hope to harmonize it with Eric Weisstein's Mathematica tools for magic squares [24].

7 Integer Points in Polyhedral Cones

In 2004 Ahmed [4] asked "How Many Squares Are There, Mr. Franklin?..." in a paper which exploited Hilbert bases of polyhedral cones (PHC) to construct several new natural Franklin squares. Ahmed [4][5] was also able to use PHC techniques to count the total number of 8th order Franklin squares as a function of a variable magic sum (s), obtaining a count of $228,881,701,845,346$ for $s = 260$. However this includes many squares with degenerate elements, so that this count is an upper bound to the population of natural squares. PHC techniques do not at present permit the elements to be distinct so that they do not give the smaller counts expected for natural squares, where the elements $1..n^2$ are distinct. These PHC techniques, and their upper bound statements, have received considerable attention recently [25], in publications [Ahmed [4] [5]; Ahmed, De Loera and Hemmecke [26]; Beck et al [27]; and have recently been the subject of advanced courses ([28][29]).

7.1 Comparison of Various Counts

We construct a table after the fashion of Trump [21] to collect some extant results for relevant counts and to summarize some of the major results:

order of square, n	4	5	8
natural magic sum, C_n	34	65	260
natural magic	880	$275,305,224$ (a)	$5.2210(70)\cdot 10^{54}$ (b)
associative natural	48	$48,544$ (b)	$2.5228(14)\cdot 10^{27}$ (b)
natural panmagic	48	$3,600$ (b)	(c)
'complete'	48	-	$368,640$ (d)
natural Franklin	0 (e)	-	$1,105,920$ (here)
PHC UPPER BOUNDS			
pan Franklin (f,g)	-	-	$10,308,923,109,408$
Franklin (f,g)	-	-	$228,881,701,845,346$
magic (h,i)	$163,890,864$		
panmagic (g)	$35,208$	$53,852,072,626$	

Table 1: Comparison of Various Counts: footnotes: (a) Schroeppel [19]; (b) Trump [21] with statistical errors for $n = 8$; (c) lies between preceding numbers in this column (b); (d) Ollerenshaw and Brée [7]; (e) Pasles [3], see his footnote 22 on page 506; (f) Ahmed [4]; (g) Ahmed [5]; (h) Ahmed, De Loera and Hemmecke [26]; (i) Beck [27].

8 Magic Squares and CSP's (Constraint Satisfaction Problems)

CSP's fall into several classes: genetic algorithms, evolutionary computing, ... Magic squares are frequently used as test targets in benchmarking improved CSP strategies. Sometimes it is the time to find the first square [30], or the time to find a complete set such as the 880 fourth order squares. The highly constrained eighth order most-perfect squares have also been used.

9 Compound Squares

An undergraduate project with Wayne Chan took an old Chinese idea for compounding a 3×3 magic square with itself to construct a 9×9 magic square, or a 3×3 with a 4×4 to make a pair of 12×12 magic squares, and devised a computer program [31] to extend this to very large squares. One of the squares is used as a frame and the other is incremented on placement in the appropriate position in the frame. Compounding preserves the row, column and pandiagonal characteristics which are common to both squares, and even for the smallest case of 9th order there are very large numbers of distinct squares [37]. As a result we were able to set a new world record sized magic square at $12,544 \times 12,544$. Since it is difficult to grasp a square of this size with numbers running from $1..157,351,936$ we used a colour scale to make an image which might pass for a piece of art [32]. In 2006 the world record for magic squares is still held by a smaller square of order 3001, because of rules which require writing or printing out the square on paper. We stopped at order $12,544$ simply because it was the largest which we could write onto a CD. It is clear that larger magic squares could be stored on a DVD, or some higher capacity disks, but there is little point since potential applications will need fast access.

Let us note that Sudoku is a special type of Latin square which bears some similarity to compound squares [33].

9.1 Multimagic Squares

Christian Boyer [34] has drawn attention to magic squares which remain magic when their elements are raised to integer powers (2 bimagic; 3: trimagic; etc.). An interesting mathematical analysis of multimagic squares has been given recently by Derksen, Eggermont, and van den Essen [35], which includes a contribution to compound squares. Rempel, Chan and Loly have a related project underway [36].

10 A Mechanical Application - Moment of Inertia

Another success growing out of teaching undergraduate classical mechanics for many years was the discovery of a new invariance, or universal property, for magic squares through calculations of their "moments of inertia" (essentially the inertia of the square through an axis perpendicular to its centre), which eventually turned out to depend only on the order of the square, i.e. the number of rows or columns [38]. The numbers in the magic square are replaced by corresponding multiples of a unit mass placed on a square unit lattice. In fact this was truly a "Eureka" moment for quite spontaneously I had the idea which fused long activity in teaching moment of inertia in introductory courses with a more recent activity in magic squares.

The moment of inertia, I_n, of a magic square of order n about an axis perpendicular to its centre is obtained by summing mr^2 for each cell, where m is the number centred in a cell and r is the distance of the centre of that cell from the centre of the square measured in units of the nearest neighbour distance. For the Lo-shu square the corner cells then have their centres at a distance of $\sqrt{2}$ from the axis. We can now calculate the sum for the 3×3:

$$I_3 = [1+3+7+9]\,(1)^2 + [2+4+6+8]\left(\sqrt{2}\right)^2 = 60 \qquad (7)$$

The moments of inertia about the horizontal and vertical axes through the centre are each 30, reminding us of the perpendicular axis theorem, which says that their sum gives the moment of inertia about the axis through the centre and perpendicular to the plane.

When I used a data file for the complete set of the 4×4's (by courtesy of Harvey Heinz [39]) it was a surprise to find that they all gave $I_4 = 340$. I was then motivated to attempt a derivation, which was easy since the calculations only depended on the semi-magic property so that the parallel axis theorem and the perpendicular axis theorem could be used. In retrospect this could have been set as an examination problem in a sophomore course on classical mechanics!

$$I_n = \frac{1}{12} n^2 (n^4 - 1) \tag{8}$$

This remarkably simple formula recovers the results for $n = 3, 4$ above and is valid for arbitrary order. The derivation of the formula only depends on the row and column properties, and not on the diagonals of magic squares, so that it actually applies to the larger class of semi-magic squares which lack one or both diagonal magic sums of magic squares.

Of related interest, Abiyev et al [40] have studied the centre of mass of certain magic squares and suggest applications to robotics.

10.1 RC (RowColumn) symmetry.

In the above one can factor out the magic linesum (C_n):

$$I_n = \frac{1}{12} n^2 (n^4 - 1) = \left[\frac{1}{2} n \left(n^2 + 1\right)\right] \left[\frac{1}{6} n \left(n^2 - 1\right)\right] = C_n \frac{1}{6} n \left(n^2 - 1\right) \tag{9}$$

for a result which applies to any square of equally spaced rows and columns with the same mass, C_n. The limit of a uniform continuous sheet agrees with the standard result using the calculus. It is clear that large random semimagic squares tend towards the limitting value of this expression.

11 Magical Cubes and Hypercubes

Along with magic squares, there has long been an interest in magic cubes, going back at least to Leibniz [41], and then later in 4D hypercubes following Riemann's n-dimensional geometry in the second half of the 19th century. There are now studies of higher dimensional hypercubes. For magic cubes there are some fascinating applications. We give a link to an image of a magic cube with spheres sized according to mass [43], as well as a paper [44] on perfect magic cubes, which also has an image.

Adam Rogers has recently helped extend my inertia ideas ideas to the calculation of the full inertia tensor of magic cubes [42]. RCP (RowColumnPillar) symmetry means that any cube of equally spaced RCP's with the same mass in each will have the same form of inertia tensor. This work has also recently been reviewed by Ivars Peterson [46].

12 Electric Quadrupoles

A new magic square topic has just emerged from my renewed involvement with our honours electromagnetism course.

The idea is to treat the numerical value of each element as an electric charge. It is soon clear for small squares that the dipole moment vanishes, so we then proceed to study the quadrupole moment. As a first thought I neutralized magic squares so that their elements ran from $-\left(n^2-1\right)/2$ to $+\left(n^2-1\right)/2$, but later Rogers and Loly [45] analyzed the multipole expansion for a normal magic square, finding that it takes care of many of the details. The full story involves calculating the quadrupole tensor, something beyond the scope of the present article. The short story is that the quadrupole tensor vanishes, so one then proceeds to the octupole!

13 Pandiagonal Non-Magic Squares (order $2^n = 2, 4, 8, 16, ...$)

Here the highlight is the discovery [51] of a small new class of purely pandiagonal non-magic, number squares having dimensions of the powers of 2, with the additional property that the binary representation changes by just 1-bit in horizontal and vertical moves (they are counted from 0..15 in the example below):

0	1	3	2
4	5	7	6
12	13	15	14
8	9	11	10

(10)

These squares derive from the Myers-Briggs dichotomous scheme of personality types. Then work with summer undergraduate Marcus Steeds (Loly and Steeds 2003), who devised a number of useful test tools, explored how generalizations of these squares are related to the Gray code and square Karnaugh maps of digital logic design, a connection made by some of my third year engineering students a few years ago. I have also applied these ideas to ancient Chinese patterns based on the yin-yang duality [52].

Recently Dan Schindel found that of the $3,465,216$ pandiagonal non-magic squares, 48 have the 1-bit property. This agrees with a symmetry argument made recently by Ian Cameron [47].

Meine and Schuett [48] of Siemens have suggested applications of these squares to cryptography and image processing [49].

14 From Coupled Oscillators to Modern Linear Algebra

Undergraduate physics provides several examples in mechanics and wave motion (from coupled oscillators) where semi-magic matrices arise with algebraic or non-integer elements. The investigation of the mechanical problems had a vibrant interplay with mathematics for two centuries from the time of Huyghens and

Newton. Huyghens, of course, is well known for his study of the isochrony of pendulum motion. An excellent chronology is found in Brillouin [53], whose study of waves in periodic systems is a tour-de-force.

A central mathematical theme in physical science and engineering concerns what are known as eigenvalue problems. These involve homogeneous linear equations which only have non-trivial solutions if the determinant of the coefficients vanishes, with as many solutions as the number of equations. These issues can be clarified by using a specific example for which the coupled oscillator is ideal. At the same time we can prepare the ground for studying magic square matrices in their own right.

14.1 Homogeneous Simultaneous Equations for the Coupled Oscillator

In the usual description of this one-dimensional problem [54] the equations of motion for masses M displaced along the x-direction from their equilibrium positions by x_1 and x_2, a coupling spring of force constant γ, and with each tied to fixed posts at opposite ends by springs of force constant κ are:

$$M\ddot{x}_1 + (\kappa+\gamma)x_1 - \gamma x_2 = 0 \tag{11}$$

$$M\ddot{x}_2 + (\kappa+\gamma)x_2 - \gamma x_1 = 0 \tag{12}$$

These are simplified by taking out a simple time-dependence: $x(t) = B\exp{(it\omega)}$ for:

$$(\kappa+\gamma-M\omega^2)B_1 - \gamma B_2 = 0 \tag{13}$$

$$-\gamma B_1 + (\kappa+\gamma-M\omega^2)B_2 = 0 \tag{14}$$

Instead of the general approach of setting the determinant of the coefficients of these simultaneous equations to zero, this simple problem may be solved simply by forming ratios of the variables:

$$\frac{B_1}{B_2} = \frac{\gamma}{(\kappa+\gamma)-M\omega^2} = \frac{(\kappa+\gamma)-M\ \omega^2}{\gamma} \tag{15}$$

Cross multiplication of the second equality results in a quadratic equation in ω^2, with two solutions, one, $\omega^2 = \frac{\kappa}{M}$, just the frequency of each oscillator without coupling, and the other higher, $\omega^2 = \frac{\kappa+2\gamma}{M}$. Extended to a chain of two or more alternating masses and springs, we have the origin of the gaps in the spectrum which are a characteristic feature of solid state physics. We continue this example after introducing some essential matrix operations, which are easy enough to cover in high school.

15 Determinants and Matrices

Cayley initiated matrix theory in 1846, followed by contributions from Peirce, Hamilton, Poincaré, and Sylvester. We highlight the issues of interest with respect to magic squares, the semi-magic matrices and tensors arising in mechanics, and the related interest for pandiagonal non-magic squares with a brief discussion using 2×2 matrices.

15.1 Matrix Multiplication

Consider the usual matrix-vector multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} aP + bQ \\ cP + dQ \end{bmatrix} \tag{16}$$

Clearly if $P = Q = 1$, this sums the rows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a + b \\ c + d \end{bmatrix} \tag{17}$$

This $[1, 1]$ vector will be referred to as a diagonal (or 2-agonal) vector, and generalizes to higher orders. However if one takes the original matrix operator to act from the right onto a row vector on the left, as a ”left-hand” problem, then one finds the column sums of the original matrix:

$$\begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} Pa + Qc & Pb + Qd \end{bmatrix} \tag{18}$$

if $P = Q = 1$, this sums the columns.

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + c & b + d \end{bmatrix} \tag{19}$$

This may also be achieved by left multiplication of the transposed matrix with a row vector as illustrated next.

15.2 Eigenproblems - Eigenvectors and Eigenvalues

My interest in these issues began with an observation from experimental computing of a few different order magic squares with Mathematica during a sophomore course which I was teaching. My colleague Frank Hruska had published a relevant and stimulating paper [55], David Lavis drew my attention to left and right eigenvectors, whilst another, Joe Williams, knew from teaching linear algebra that this eigenvector adds row elements, etc. The utility of the "n-agonal" eigenvector $[1, 1, 1, ...]$ of the n-cube is seen by showing how the rows sum in the Lo-shu magic square :

$$\begin{bmatrix} 3 & 5 & 7 \\ 4 & 9 & 2 \\ 8 & 1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} = 15 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{20}$$

where we have factored out the eigenvalue, 15, to show the action of the matrix operator in leaving the eigenvector unchanged. The columns have the same eigenvalue as follows from the transposed matrix:

$$\begin{bmatrix} 3 & 4 & 8 \\ 5 & 9 & 1 \\ 7 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} = 15 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (21)$$

Alternatively we can work with a row eigenvector on the left with the original matrix:

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \left[\begin{bmatrix} 3 & 5 & 7 \\ 4 & 9 & 2 \\ 8 & 1 & 6 \end{bmatrix} \right] = \begin{bmatrix} 15 & 15 & 15 \end{bmatrix} = 15 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \quad (22)$$

We must note that the n-agonal eigenvector is both a left and a right eigenvector, and that this property depends only on the semi-magic property. An immediate application is now afforded by the coupled oscillator.

16 Coupled Oscillator Eigenvectors

When written in modern matrix notation, the semi-magic nature of the characteristic equation $(13, 14)$ is apparent:

$$\begin{bmatrix} \kappa + \gamma & -\gamma \\ -\gamma & \kappa + \gamma \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = M\omega^2 \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad (23)$$

It is immediately clear that a solution is the $[1, 1]$ eigenvector, both as a right, and as a left eigenvector. It has the lower eigenvalue of $\omega^2 = \frac{\kappa}{M}$. The other eigenvector is $[1, -1]$, which corresponds to the higher solution of $\omega^2 = \frac{\kappa+2\gamma}{M}$

A similar semi-magic property is also found for the full moment of inertia tensor of magic cubes [42].

16.1 Left and right eigenvectors

Of significant interest for our studies of magic squares, this topic is important for teaching linear algebra beyond the introductory course. The use of magic square examples already occurs in such courses, but it can be used even more seriously. The existence of identical left and right eigenvectors implies deeper properties giving rise to the theorem of biorthogonality, and the theorem of Perron [56]. These theorems develop deep links between the left and right eigenvectors and eigenvalues. As such, magic squares are insightful examples for advanced linear algebra courses.

Software such as Mathematica, Maple and MATLAB can be profitably employed in such studies. Indeed, MATLAB has initiated some of this already by including a function, *magic*(n), which returns a magic square from one of three algorithms, one each for odd, even and doubly-even cases. A drawback with $magic(n)$ is that the single squares which result are not representative of the richness of the spectrum of magic squares of a given order, save for $n > 3$.

16.2 Eigenvalues of Integer Squares

In another study we have nearly finished a study of the eigenvalues of magic square matrices [58]. For the 880 distinct 4×4's in the 12 Dudeney groups, we find that members of the first six (singular) groups have three distinct eigenvalue patterns, with a subset of the first three groups having three zero eigenvalues, while the last six (non-singular) groups have two further eigenvalue patterns. Also if the 1-bit pandiagonal non-magic squares discussed earlier are treated as matrices they possess examples with just two non-zero eigenvalues for any order (Loly and Steeds, 2003).

I can add here that 8th order Franklin squares have 3 non-zero eigenvalues, as do also the corresponding most-perfect squares.

Stephen Wolfram was interested in these large nullspaces, and in what one might conclude about random matrices.

17 Conclusion

Further information on the history may be found in recent books (Swetz, 2000) by Frank Swetz, a mathematics educator, by René Descombes (2001), as well as in Clifford Pickover's recent book (Pickover, 2002). Those sources also enlarge on the philosophical aspects, which began in China as a cosmology, or organizing scheme.

There are opportunities to enrich teaching in classical physics, and likely in quantum physics as well. Certainly more can be done in the context of teaching linear algebra, which can begin in high school. I have found wonderful opportunities for students to cooperate in some group work as summer research assistants, indeed their enthusiasm and initiative in tackling problems has been gratifying.

18 Acknowledgments

This article is an elaboration of a poster talk given at NKS2006, which has been augmented as a result of feedback from Stephen Wolfram and others. David Lavis, Frank Hruska, Joe Williams, Walter Trump, John Hendricks, and Harvey Heinz have helped with advice on various aspects of this study, which engaged the following undergraduates: Marcus Steeds, Wayne Chan, Adam Rogers, Matt Rempel and Dan Schindel.

Critical funding for this research was provided in 2003 by the Winnipeg Foundation Post Secondary Grants Program, especially for support to Adam Rogers and Matt Rempel. My colleagues Professors John Vail and Tom Osborn have generously let their summer student Daniel Schindel continue working part time on some of these projects during 2003 and 2004.

References

[1] Schindel, D., M. Rempel and Loly, P.D., 2006, Enumerating the bent diagonal squares of Dr Benjamin Franklin FRS, Proc. R. Soc. A: Physical, Mathematical and Engineering, 462, 2271-2279, August 2006.

[2] Amela, M., Structured 8 x 8 Franklin Squares, http://www.region.com.ar/amela/franklinsquares/

[3] Pasles, P. C., The lost squares of Dr. Franklin: Ben Franklin's missing squares and the secret of the magic circle, Am. Math. Monthly, 108 (2001), pp. 489-511.

[4] Ahmed, M. M., How Many Squares Are There, Mr. Franklin?: Constructing and Enumerating Franklin Squares, Am. Math. Monthly, 111 (2004), pp. 394-410.

[5] Ahmed, M.M. 2004 Algebraic Combinatorics of Magic Squares, Ph.D. dissertation, University of California, Davis..

[6] Peterson, I., Counting Franklin's Magic Squares, Science News Online, 24 June 2006, http://www.sciencenews.org/articles/20060624/mathtrek.asp

[7] Ollerenshaw, K. and Brée, D. S.: 1998, Most-perfect pandiagonal magic squares: their construction and enumeration. Southend-on-Sea, U.K.: The Institute of Mathematics and its Applications.

[8] Swetz, F. J.: 2002, Legacy of the Luoshu - The 4000 Year Search for the Meaning of the Magic Square of Order Three. Chicago: Open Court.

[9] Pickover, C.: 2002, The Zen of Magic Squares, Circles, and Stars - An Exhibition of Surprising Structures across Dimensions. Princeton University Press.

[10] Frénicle de Bessy, M., Des Quarrez ou Tables Magiques, Histoire de l'Académie royale des sciences avec les mémoires de mathématique et de physique tirés des registres de cette Académie, 1729 (T.4), p.209-304; http://gallica.bnf.fr/ark:/12148/bpt6k34948

[11] Frénicle de Bessy, M., Table Generale des Quarrez Magiques de quatre, Histoire de l'Académie royale des sciences avec les mémoires de mathématique et de physique tirés des registres de cette Académie, 1729 (T.4), p.305-354; http://gallica.bnf.fr/ark:/12148/bpt6k34948

[12] Bragdon, C.,The Frozen Fountain: Being Essays on Architecture and the Art of Design in Space, New York, 1932, Alfred A. Knopf, Kessinger Publishing's Rare Mystical Reprints, www.kessinger.net

[13] Myers, I. B. with P. B. Myers, 1980, 1993. Gifts Differing: Understanding Personality Type. CPP Books, Palo Alto, California.

[14] von Franz, M. L.: 1974, Number and Time: Reflections Leading Toward a Unification of Depth Psychology and Physics, Northwestern University Press.

[15] McClintock, E., On the Most Perfect Forms of Magic Squares, with Methods for Their Production, Am. J. Math. 19 (1897), pp. 99-120.

[16] WO 99/66671 Patent: Ollerenshaw, Kathleen and Brée, David S., Transposition Cipher, pp. 30, 17 June 1999.

[17] Besslich, Ph. W., Comments on electronic techniques for pictorial image reproduction, IEEE Trans. on Comm. 31, 846-847, 1983

[18] Besslich, Ph. W., A Method for the Generation and Processing of Dyadic Indexed Data, IEEE Trans. on Computers, C-32, No. 5, 487-4494, 1983

[19] Schroeppel, R.: 1973, see M. Gardner in his Scientific American column, Jan. 1976.

[20] Pinn, K. and Wieczerkowski, C.: 1998, Number of Magic Squares from Parallel Tempering Monte Carlo. International Journal of Modern Physics C, 9, No. 4, 541-546.

[21] Trump, W.: 2003, *Notes on Magic Squares and Cubes*, http://www.trump.de/magic-squares/

[22] Meyer, H. B., *magic squares of order 4*, www.faust.fr.bw.schule.de/mhb/backtrack/mag4en.htm

[23] Loly, P. D. and Schindel, D. G., A simplified demonstration of counting the 880 fourth order magic squares using Mathematica [a draft Notebook]

[24] Weisstein, E., Magic Square, Wolfram MathWorld, http://mathworld.wolfram.com/MagicSquare.html

[25] D. Mackenzie, Quadramagicology, New Scientist, 180 (2003), pp. 50-53.

[26] M. M. Ahmed, J. De Loera, and R. Hemmeke, Polyhedral cones of magic cubes and squares, in New Directions in Computational Geometry, The Goodman-Pollack Festschrift volume, Aronov et al., eds., Springer-Verlag, 2003, pp. 25-51.

[27] M. Beck, M. Cohen, J. Cuomo, and P. Gribelyuk, The number of "magic" squares and hypercubes, Amer. Math. Monthly 110 (2003), pp. 707-717.

[28] Joint AMS-IMS-SIAM Summer Research Conference on Integer Points in Polyhedra, Geometry, Number Theory, Algebra, Optimization, 2003; proceedings to appear in Cont. Math. (Proceedings of the Summer Research Conference on Integer Points in Polyhedra, July 13 - July 17, 2003 in Snowbird, Utah).: ams.org/meetings/src-barvin.html

[29] Banff International Research Station summer school, 2005: pims.math.ca/birs/workshops/2005/05ss027/

[30] Jacobs, C. and Hadler, H., TITLE, http://ls11-www.cs.uni-dortmund.de/people/kursawe/Demos/EvoNet/index.html

[31] Chan, W. and Loly, P. D.: 2002, Iterative Compounding of Square Matrices to Generate Large-Order Magic Squares, Mathematics Today (IMA), 38(4), 113-118.

[32] Chan, W. S. and Loly, P. D., Colour image of 12,544th order compound magic square, http://home.cc.umanitoba.ca/~loly/smallmath.jpg

[33] Pegg, E. Jr., Sudoku, Wolfram MathWorld, http://mathworld.wolfram.com/Sudoku.html

[34] Boyer,C., Multimagic squares, http://www.multimagie.com/

[35] Derksen, H., Eggermont, C. and van den Essen, A., Multimagic Squares, arXiv:math.CO/0504083 v2 5 Apr 2005, to appear in Am. Math. Mnthly.

[36] Rempel, M., Chan, W.S. and Loly, P.D., Compounding Regular and Multimagic Squares, and a Second Method, preprint.

[37] Bellew, James (1997). Counting the Number of Compound and Nasik Magic Squares, Mathematics Today, August 1997, 111–118.

[38] Loly, P. D.: 2004, The Invariance of the Moment of Inertia of Magic Squares. The Mathematical Gazette 88 (511), 151-3, March 2004.

[39] Heinz, H., Order-4 Magic Squares, http://www.geocities.com/~harveyh/order4list.htm

[40] Abiyev, A-A, et al, Investigation of center of mass by using magic squares and its possible engineering applications, Robotics and Autonomous Systems, 49 (2004) 219-226.

[41] Heinz, H., Magic Cubes - Introduction, http://members.shaw.ca/hdhcubes/

[42] Rogers, A. and Loly, P. D.: 2004 The Inertia Tensor of Magic Cubes, American Journal of Physics, 72(6), 786-9, June 2004

[43] Loly, P. D., Magic Cube with Massive Balls using Visual Python, http://home.cc.umanitoba.ca/~loly/squareBalls6.bmp

[44] Boyer, C., Perfect magic cubes, http://cboyer.club.fr/multimagie//English/Perfectcubes.htm

[45] Rogers, A. and Loly, P. D.: 2005 The electric multipole expansion for a magic cube, European Journal of Physics 26 (2005) 809-813.

[46] Peterson, I., Magic Square Physics, Science News Online, 1 July 2006, http://www.sciencenews.org/articles/20060701/mathtrek.asp

[47] Cameron, I.: 2003, private communication.

[48] Meine, S., and Schuett, D., On Karnaugh Maps and Magic Squares, Informatik-Spektrum, 28(2), 120-123 (2005).

[49] Schuett, D., Eckhardt, U. and Suda, P., Boot Algebras, TR-92-039, Jun 1992, International Computer Science Institute, Berkeley, CA.

[50] Loly, P. D.: 2002-2003, A purely pandiagonal 4*4 square and the Myers-Briggs type Table, J. Rec. Math., 31 (1), 29-31.

[51] Loly, P. D. and Steeds, M.: 2002, A New Class of Pandiagonal Non-Magic Squares, Int. J. Math. Ed. Sci. Tech. 36 (4), 2005, 375-388,

[52] Loly, P. D.: 2002, A Logical Way of Ordering the Trigrams and Hexagrams of the Yijing. The Oracle: The Journal of Yijing Studies, 2 (12), 2-13.

[53] Brillouin, L.: 1946, Wave Propagation in Periodic Structures. New York: McGraw-Hill.

[54] Marion, J. B. and Thornton, R. K.: 1995, Classical Dynamics of Particles and Systems, 4th edition, Harcourt, Brace, Jovanovich.

[55] Hruska, F.: 1991, Magic Squares, Matrices, Planes and Angles. J. Rec. Math. 23(3), 183-189.

[56] Mattingly, R. B.: 2000, Even Order Regular Magic Squares are Singular. American Mathematical Monthly, 107, 777.

[57] Moler, C., MATLAB's Magical Mystery Tour, The MathWorks Newsletter, 7 (1), 8-9, 1993.

[58] Schindel, D.S., Trump, W. and Loly, P. D., Singular Regular Magic Squares: a special set of skew-centrosymmetric matrices, preprint 2006

[59] Descombes, R., 2001: Les Carrés Magiques, Histoire, théorie et technique du carré magique, de l'Antiquité aux recherches actuelles. Vuibert, Paris.